Rare Disaster Probability and Options Pricing *

Robert J. Barro
Harvard University

Gordon Y. Liao
Federal Reserve Board

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Abstract

We derive an option-pricing formula from recursive preferences and estimate rare disaster probability. The new options-pricing formula applies to far-out-of-the money put options on the stock market when disaster risk dominates, the size distribution of disasters follows a power law, and the economy has a representative agent with a constant-relative-risk-aversion utility function. The formula conforms with options data on the S&P 500 index from 1983-2018 and for analogous indices for other countries. The disaster probability, inferred from monthly fixed effects, is highly correlated across countries, peaks during the 2008-2009 financial crisis, and forecasts rates of economic growth.

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** Corresponding author.

Email addresses: rbarro@harvard.edu (R. Barro), gordon.y.liao@frb.gov (G. Liao).

The options market provides rich information on the forward-looking distribution of asset returns. Translating security prices into the objective distribution of economic growth, however, requires specifying a model or a set of recovery assumptions. While past research has studied tail outcomes priced by equity options, the existing methods often require risk neutrality, involved estimation techniques, and high-frequency data. These challenges make robust estimation and use of disaster probability series difficult.

We derive a tractable options-pricing model that applies when disaster risk is the dominant force and extract disaster probability series with applications in several settings. The model assumes constant relative risk aversion and a power-law form for disaster sizes. We assess within this model the pricing of far-out-of-the-money put options on the overall stock market, corresponding empirically to the S&P 500 in the United States and analogous indices for other countries. The simple pricing formula applies when the option is sufficiently far out of the money (operationally, a relative exercise price or moneyness of 0.9 or less) and when the maturity is not too long (operationally, up to 6 months).

In the prescribed region, the elasticity of the put-options price with respect to maturity is close to one. The elasticity with respect to the exercise price is greater than one, roughly constant, and depends on the difference between the power-law tail parameter, denoted α, and the coefficient of relative risk aversion, γ. We show that the theoretical formula conforms with data from 1983 to 2018 on far-out-of-the-money put options on the U.S. stock market and analogous indices over shorter periods for other countries.

The options-pricing formula involves a term that is proportional to the disaster probability, p. This term depends also on three other parameters: γ, α, and the threshold disaster size, z_0. If these three parameters are fixed, we can use estimated time fixed effects to gauge the time variations in p. The options-pricing formula depends also on potential changes in p. Specifically, sharp increases in p can get out-of-the-money put options into the money without the realization of a disaster. We find empirically that the probability, q, of a large upward movement in p can be treated as roughly constant.

Relative to the standard approach in the literature that assumes a risk-neutral distribution, our approach starts with the preferences of a representative investor. This preference-based approach intuitively connects option pricing with consumption and growth. The two approaches are, in fact, similar because our disaster-probability time series is jointly estimated with the degree of risk aversion, γ, and the power law parameter, α. However, even though time variations in γ and α cannot be pinned down separately, the unconditional estimates of γ and α, derived from the estimated elasticity of options price with respect to exercise price, accords with estimates from previous estimates using macro variables.

This market-based assessment of objective disaster probability provides a valuable indicator of tail risks in the aggregate economy. The disaster probability p is highly correlated across countries and varies significantly over time. We use p to forecast growth vulnerabilities – defined as GDP growth at the lowest decile. An increase in disaster probability is associated with a decline in the conditional mean of growth – downside risks to growth rise with disaster probability while upside risks are independent of disaster probability. Moreover, disaster probability as registered by the financial markets contains different information about tail risks in the economy when compared to political uncertainty.

I. Related Literature

Our model belongs to the class of jump-diffusion models. Options pricing within this general class goes back to Merton (1976) and Cox and Ross (1976). More recently, empirical estimation and validation of jump-diffusion models have been conducted under different contexts. Bates (2006) develops a maximum-likelihood methodology for estimating latent affine processes. Santa-Clara and Yan (2010) build a linear-quadratic jump-diffusion model and use it to separate diffusion and jump processes. Relative to earlier studies, we incorporate rare disaster risk in a preference-based model that relates option prices to consumption rare disaster risk and delivers a simple closed-form formula that conforms with data.

A number of papers have examined the variance risk premium and realized volatility. In particular, Andersen et al. (2003) build a forecasting model of realized volatility using intraday data. Bollerslev, Tauchen, and Zhou (2009) study the predictability of the aggregate stock return using variance risk premia. Londono and Xu (2019) study the downside and upside variance risk premium and their predictive powers for international stock returns. Relative to these papers, we focus on the disaster component of the volatility or variance risk premium.

The use of far-out-of-the-money put option prices to infer disaster probabilities was pioneered by Bates (1991). This idea has been applied recently by, among others, Bollerslev and Todorov (2011a, 2011b); Backus, Chernov, and Martin (2011); Seo and Wachter (2016); Ross (2015); and Siriwardane (2015). In particular, Bollerslev and Todorov (2011b) estimate jump risk using high-frequency data and find that compensation for rare events accounts for a large fraction of average equity and variance risk premia. In contrast, Backus, Chernov, and Martin (2011) find that option implied probabilities of rare events are smaller than those estimated from macroeconomic data. Seo and Wachter (2016) reconcile the findings by allowing disaster probability to be stochastic. Gabaix (2012) explains a number of asset-pricing puzzles including high put options prices with rare disaster risk by applying linearity-generating processes and incorporating time-varying disaster sensitivity. Our approach models time-varying disaster probability in a tractable formula derived from recursive preferences. One advantage of our method is the convenience it provides for estimating disaster probability using low-frequency options data. Supplementing the model with a rich data set of international equity index options, we also contribute to the literature by providing time series estimates of disaster probabilities for a number of countries.

The application of our estimated disaster probability to forecasting economic growth vulnerabilities echoes the work of Adrian, Boyarchenko, and Giannone (2019), which relates the conditional distribution of GDP growth to a financial-conditions index. In this study, we show that disaster risk, extracted from market prices, is an important component of financial conditions and determinants of growth vulnerabilities.

Part II lays out the rare-disasters framework. Part III works out a formula for pricing of put options within the disaster setting. The analysis starts with a constant probability, p, of disasters and then introduces possibilities for changing p_t. Part IV sets up the empirical framework, describes the data and the fit of the model, and discusses the application of the estimated p_t to forecasting growth vulnerabilities. Part V concludes.

II. Rare-Disaster Model and Previous Results

We use a familiar setup based on rare-macroeconomic disasters, as developed in Rietz (1988) and Barro (2006, 2009). The model is set up for convenience in discrete time. Real GDP, Y, is generated from

$$(1) \qquad log(Y_{t+1}) = log(Y_t) + g + u_{t+1} + v_{t+1},$$

where, $g \geq 0$ is the deterministic part of growth, u_{t+1} (the diffusion term) is an i.i.d. normal shock with mean 0 and variance σ^2, and v_{t+1} (the jump term) is a disaster shock. Disasters arise from a Poisson process with probability of occurrence p per period. For now, p is taken as constant. A later section allows for time variations in p, which then play a central role. When a disaster occurs, GDP falls by the fraction b, where $0 < b \leq 1$. The distribution of disaster sizes is time invariant. (The baseline model includes disasters but not bonanzas.) This jump-diffusion process for GDP is analogous to the one posited for stock prices in Merton (1976, equations [1]-[3]).[1]

In the underlying Lucas (1978)-tree model, which assumes a closed economy, no investment, and no government purchases, consumption, C_t, equals GDP, Y_t. The implied expected growth rate of C and Y is given, if the period length is short, by

(2) $$g^* = g + (1/2) \cdot \sigma^2 - p \cdot E[b].$$

In this and subsequent formulas, we use an equal sign, rather than approximately equal, when the equality holds as the period length shrinks to zero.

The representative agent has Epstein-Zin/Weil utility,[2] as in Barro (2009):

(3) $$[(1-\gamma)U_t]^{\left(\frac{1-\theta}{1-\gamma}\right)} = C_t^{1-\theta} + \left(\frac{1}{1+\rho}\right) \cdot [(1-\gamma)E_t U_{t+1}]^{\left(\frac{1-\theta}{1-\gamma}\right)},$$

where $\gamma > 0$ is the coefficient of relative risk aversion, $\theta > 0$ is the reciprocal of the intertemporal-elasticity-of-substitution (IES) for consumption, and $\rho > 0$ is the rate of time preference. As shown in Barro (2009) (based on Giovannini and Weil [1989] and Obstfeld [1994]), with i.i.d. shocks and a representative agent, the attained utility ends up satisfying the form:

(4) $$U_t = \Phi \cdot C_t^{1-\gamma}/(1-\gamma),$$

[1]Related jump-diffusion models appear in Cox and Ross (1976).
[2]Epstein and Zin (1989) and Weil (1990).

where the constant $\Phi > 0$ depends on the parameters of the model. Using equations (3) and (4), the first-order condition for optimal consumption over time follows from a perturbation argument as

$$(5) \qquad \left[E_t\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma}\right]^{\left(\frac{\gamma-\theta}{\gamma-1}\right)} = \left(\frac{1}{1+\rho}\right) \cdot E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \cdot R_{t+1}\right],$$

where R_{t+1} is the gross rate of return on any available asset from time t to time $t+1$. When $\gamma = \theta$—the familiar setting with time-separable power utility—the term on the left-hand side of equation (5) equals one.

The process for R and Y in equation (1) implies, if the period length is negligible:

$$(6) \qquad E_t\left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} = 1 + (1-\gamma)g - p + p \cdot E(1-b)^{1-\gamma} + \left(\tfrac{1}{2}\right)(1-\gamma)^2\sigma^2.$$

This condition can be used along with equation (5) to price various assets, including a risk-free bond and an equity claim on a perpetual flow of consumption (that is, the Lucas tree).

Equations (5) and (6) imply that the constant risk-free real interest rate is given by

$$(7) \qquad r^f = \rho + \theta g^* - p \cdot \left[E(1-b)^{-\gamma} - \left(\frac{\gamma-\theta}{\gamma-1}\right) E(1-b)^{1-\gamma} - \theta \cdot Eb + \left(\frac{1-\theta}{\gamma-1}\right)\right] - \left(\tfrac{1}{2}\right)\gamma(1+\theta)\sigma^2 .$$

Let P_t be the price at the start of period t of an unlevered equity claim on the Lucas tree. Let V_t be the dividend-price ratio; that is, the ratio of P_t to C_t. In the present model with i.i.d. shocks, V_t equals a constant, V, so that the growth rate of P_t equals the growth rate of C_t. The reciprocal of V equals the dividend-price ratio and can be determined from equations (5) and (6) to be

$$(8) \qquad \frac{1}{V} = \rho - (1-\theta)g^* + p \cdot \left[\left(\frac{1-\theta}{\gamma-1}\right) E(1-b)^{1-\gamma} - (1-\theta) \cdot Eb - \left(\frac{1-\theta}{\gamma-1}\right)\right] + \left(\tfrac{1}{2}\right)\gamma(1-\theta)\sigma^2.$$

The constant expected rate of return on equity, r^e, is the sum of the dividend yield, $1/V$, and the expected rate of capital gain on equity, which equals g^*, the expected growth rate of the dividend (consumption). Therefore, r^e is the same as equation (8) except for the elimination of the term $-g^*$. (The transversality condition, which ensures that the value of tree equity is positive and finite, is $r^e > g^*$.) The constant equity premium is given from equations (7) and (8) by:

(9) $$r^e - r^f = \gamma\sigma^2 + p \cdot [E(1-b)^{-\gamma} - E(1-b)^{1-\gamma} - Eb].$$

The disaster or jump term in equation (9) is proportional to the disaster probability, p. The expression in brackets that multiplies p depends on the size distribution of disasters, b, and the coefficient of relative risk aversion, γ. These effects were calibrated in Barro (2006) and Barro and Ursúa (2012) by using the long-term history of macroeconomic disasters for 40 countries to pin down p and the distribution of b. The results accord with an observed average unlevered equity premium of 0.04-0.05 per year if γ is around 3-4.

The diffusion term, $\gamma\sigma^2$, in equation (9) is analogous to the expression for the equity premium in Mehra and Prescott (1985) and is negligible compared to the observed average equity premium if γ and σ^2 take on empirically reasonable values. For many purposes—including the pricing of far-out-of-the-money stock options—this term can be ignored.

III. Pricing Stock Options

We now discuss the pricing of stock options within our model, which fits into the class of jump-diffusion models. There is a long-standing literature on options-pricing models with jump risk. Our preference-based approach excels in its simplicity and ability to connect market prices to macroeconomic variables. The approach is also useful to assess state prices under the physical measure, separating the risk premium from objective expectations of outcomes. The merits of the preference approach come at a cost of requiring assumptions on preferences. For instance, our model assumes a time-invariant constant-relative-risk-aversion utility function.

We first consider the case in which the representative agent perceives the probability of disaster to be constant. This baseline framework is presented in sections A-D below. Under this counterfactual scenario, the time-variations in option prices can only be generated if the agent reprices options in each period with a new probability and assumes that the new probability would hold indefinitely. This type of assumption has been used in the literature, for instance, in Weitzman (2007), Bakshi and Skoulakis (2010) and Cogley and Sargent (2008). However, as options traders do anticipate changes in the probability of disaster, our baseline model generates options prices that are lower than traded options prices, or alternatively, the disaster probability implied from traded options using the baseline model is upward-biased. To correct for this mis-specification, we introduce, in section F an assumption on the process that generates changes in disaster probability. This framework allows for a time-fixed-effects procedure to back out a time series for disaster probability.

A. Setup for pricing options

We derive a pricing solution for far-out-of-the-money put options under the assumption that disaster events (jumps) are the dominant force. Key underlying conditions for the validity of the solution are that the option be sufficiently far out of the money and that the maturity is not too long. Under these conditions, we derive a simple pricing formula that reflects the underlying Poisson nature of disaster events, combined with an assumed power-law distribution for the sizes of disasters. This formula generates testable hypotheses—which we subsequently test—on the relation of the put-options price to maturity and exercise price. Furthermore, the linearity of the formula in disaster probability and the constant density function characterizing disaster size allow the identification of the physical probability of disaster.

Consider a put option on equity in the Lucas tree. To begin, suppose that the option has a maturity of one period and can be exercised only at the end of the period (a European option). The exercise price or strike on the put option is

(10) $$exercise\ price = \varepsilon \cdot P_t,$$

where we assume in the main analysis that $0 < \varepsilon \leq 1$. We refer to ε, the ratio of the exercise price to the stock price, as the relative exercise price (often described as "moneyness").

The payoff on the put option at the start of period $t+1$ is zero if $P_{t+1} \geq \varepsilon \cdot P_t$. If $P_{t+1} < \varepsilon \cdot P_t$, the payoff is $\varepsilon P_t - P_{t+1}$. If $\varepsilon < 1$, the put option is initially out of the money. We focus empirically on options that are sufficiently far out of the money (ε sufficiently below one) so that the diffusion term, u, in equation (1) has a negligible effect on the chance of getting into the money over one period. The value of the put option then hinges on the disaster term, v. Specifically, the value of the put option depends on the probability, p, of experiencing a disaster and the distribution of disaster sizes, b. Further, what will mostly matter is the likelihood of experiencing one disaster. As long as the period (the maturity of the option) is not too long, the chance of two or more disasters has a second-order pricing impact that can be ignored as a good approximation.[3]

As the diffusion term is negligible for options that are sufficiently far out of the money, the change in the stock price, P_{t+1}/P_t, reflects a disaster shock that varies in size. In a later section, we also consider time-varying disaster probability as a driver of stock-price change. In this context, the dividend-price ratio in equation (8) is constant.

[3]Similarly, if we allowed for possible bonanzas, we could neglect the chance of a disaster and a bonanza both occurring over the period.

Let the price of the put option at the start of period t be $\Omega \cdot P_t$. We refer to Ω, the ratio of the options price to the stock price, as the relative options price. The gross rate of return, R^o_{t+1}, on the put option is given by

$$(11) \qquad R^o_{t+1} = \begin{cases} 0 \ \text{ if } \ \frac{P_{t+1}}{P_t} \geq \varepsilon \\ \frac{1}{\Omega} \cdot \left(\varepsilon - \frac{P_{t+1}}{P_t}\right) \ \text{ if } \ \frac{P_{t+1}}{P_t} < \varepsilon \end{cases}.$$

If there is one disaster of size b, the put option is in the money at the start of period $t+1$ if $\frac{P_{t+1}}{P_t} = (1+g) \cdot (1-b) < \varepsilon$. We work with the transformed variable $z \equiv 1/(1-b)$, which corresponds to the ratio of normal to disaster consumption. The condition $0 < b \leq 1$ translates into $z > 1$, with z tending to infinity as b tends to 1. When expressed in terms of z, the gross rate of return on the put option is modified from equation (11) to:

$$(12) \qquad R^o_{t+1} = \begin{cases} \frac{1}{\Omega} \cdot \left(\varepsilon - \frac{1+g}{z}\right) \text{if 1 disaster occurs and } z > (1+g)/\varepsilon \\ 0 \text{ otherwise} \end{cases}.$$

To determine Ω, we use the first-order condition from equation (5), with R_{t+1} given by R^o_{t+1} from equation (12). The results depend on the form of the distribution for z, to which we now turn.

B. Power-law distribution of disaster sizes

Based on the findings for the distribution of observed macroeconomic disaster sizes in Barro and Jin (2011), we assume that the density function for z conforms to a power law:[4]

$$(13) \qquad f(z) = Az^{-(1+\alpha)}, \mathit{where}\ A > 0, \alpha > 0, \mathit{and}\ z \geq z_0 > 1\ .$$

[4]In Kou (2002, p. 1090), a power-law distribution is ruled out because the expectation of next period's asset price is infinite. This property applies because Kou allows for favorable jumps (bonanzas) and, more importantly, he assumes that the power-law shock enters directly into the log of the stock price. This problem does not arise in our context because we consider disasters and not bonanzas, and, more basically, because our power-law shock multiplies the level of GDP (and consumption and the stock price), rather than adding to the log of GDP.

This type of power law was applied by Pareto (1897) to the distribution of high incomes. The power-law distribution has since been used widely in physics, economics, computer science, and other fields. For surveys, see Mitzenmacher (2003) and Gabaix (2009), who discusses underlying growth forces that can generate power laws. Examples of applications include sizes of cities (Gabaix and Ioannides [2004]), stock-market activity (Gabaix, et al. [2003] and Plerou, et al. [2004]), CEO compensation (Gabaix and Landier [2008]), and firm size (Luttmer [2007]). The power-law distribution has been given many names, including heavy-tail distribution, Pareto distribution, Zipfian distribution, and fractal distribution.

The parameter $z_0 > 1$ in equation (13) is the threshold beyond which the power-law density applies. For example, in Barro and Ursúa (2012), the floor disaster size of $b_0 = 0.095$ corresponds to $z_0 = 1.105$. We treat z_0 as a constant. The condition that $f(z)$ integrates to one from z_0 to infinity implies $A = \alpha z_0^\alpha$. Therefore, the power-law density function in equation (13) becomes

(14) $$f(z) = \alpha z_0^\alpha \cdot z^{-(1+\alpha)},\ z \geq z_0 > 1 .$$

The key parameter in the power-law distribution is the Pareto tail exponent, α, which governs the thickness of the right tail. A smaller α implies a thicker tail.

The probability of drawing a transformed disaster size above z is given by

(15) $$1 - F(z) = (\frac{z}{z_0})^{-\alpha}.$$

Thus, the probability of seeing an extremely large transformed disaster size, z (expressed as a ratio to the threshold, z_0), declines with z in accordance with the tail exponent $\alpha > 0$.

One issue about the power-law density is that some moments related to the transformed disaster size, z, might be unbounded. For example, in equation (7), the risk-free rate depends inversely on the term $E(1-b)^{-\gamma}$. Heuristically (or exactly with time-separable power utility), we can think of this term as representing the expected marginal utility of consumption in a disaster state relative to that in a normal state. When $z \equiv 1/(1-b)$ is distributed according to $f(z)$ from equation (14), we can compute

$$(16) \qquad E(1-b)^{-\gamma} = E(z^{\gamma}) = \left(\frac{\alpha}{\alpha-\gamma}\right) \cdot z_0^{\gamma} \text{ if } \alpha > \gamma.$$

The term on the right side of equation (15) is larger when γ is larger (more risk aversion) or α is smaller (fatter tail for disasters). But, if $\alpha \leq \gamma$, the tail is fat enough, relative to the degree of risk aversion, so that the term blows up. In this case, r^f equals minus infinity in equation (7), and the equity premium is infinity in equation (9). Of course, in the data, the risk-free rate is not minus infinity and the equity premium is not infinity. Therefore, the empirical application of the power-law density in Barro and Jin (2011) confined γ to a range that avoided unbounded outcomes, given the value of α estimated from the observed distribution of disaster sizes. That is, the unknown γ had to satisfy $\gamma < \alpha$ in order for the model to have any chance to accord with observed average rates of return.[5] This condition, which we assume holds, enters into our analysis of far-out-of-the-money put-options prices.

Barro and Jin (2011, Table 1) estimated the power-law tail parameter, α, in single power-law specifications (and also considered double power laws). The estimation was based on macroeconomic disaster events of size 10% or more computed from the long history for many countries of per capita personal consumer expenditure (the available proxy for consumption, C) and per capita GDP, Y. The estimated values of α in the single power laws were 6.3, with a 95%

[5]With constant absolute risk aversion and a power-law distribution of disaster sizes, the relevant term has to blow up. The natural complement to constant absolute risk aversion is an exponential distribution of disaster sizes. In this case, the relevant term is bounded if the parameter in the exponential distribution is larger than the coefficient of absolute risk aversion. With an exponential size distribution and constant relative risk aversion, the relevant term is always finite.

confidence interval of (5.0, 8.1), for C and 6.9, with a 95% confidence interval of (5.6, 8.5), for Y.[6] Thus, the observed macroeconomic disaster sizes suggest a range for α of roughly 5-8.

C. Options-pricing formula

To get the formula for Ω, the relative options price, we use the first-order condition from equations (5) and (6), with the gross rate of return, R_{t+1}, corresponding to the return R^o_{t+1} on put options in equation (12). We can rewrite this first-order condition as

$$(17) \qquad 1+\hat{\rho} = (1+g)^{-\gamma} \cdot E_t(z^{\gamma} R^o_{t+1}),$$

where $z \equiv 1/(1-b)$ is the transformed disaster size and $1+\hat{\rho}$ is an overall discount term, given from equations (5) and (6) (when the diffusion term is negligible) by

$$(18) \qquad 1+\hat{\rho} = 1+\rho-(\gamma-\theta)g + p \cdot \left(\frac{\gamma-\theta}{\gamma-1}\right) \cdot \left[E(1-b)^{1-\gamma}-1\right] .$$

We can evaluate the right-hand side of equation (17) using the density $f(z)$ from equation (14) along with the expression for R^o_{t+1} from equation (12). The result involves integration over the interval $z \geq (1+g)/\varepsilon$ where, conditional on having one disaster, the disaster size is large enough to get the put option into the money. The formula depends also on the probability, p, of having a disaster. Specifically, we have:

$$(19) \qquad (1+\hat{\rho})(1+g)^{\gamma} = \frac{p}{\Omega} \cdot \int_{\left(\frac{1+g}{\varepsilon}\right)}^{\infty} \left\{ z^{\gamma} \cdot \left[\varepsilon - \frac{1+g}{z}\right] \cdot \alpha z_0^{\alpha} z^{-(1+\alpha)} \right\} dz .$$

Evaluating the integral[7] (assuming $\gamma < \alpha$ and $\varepsilon < [1+g]/z_0$) leads to a closed-form formula for the relative options price:

$$(20) \qquad \Omega = \frac{\alpha z_0^{\alpha}}{(1+\hat{\rho}+\alpha g)} \cdot \frac{p\varepsilon^{1+\alpha-\gamma}}{(\alpha-\gamma)(1+\alpha-\gamma)} .$$

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[6]Barro and Jin (2011, Table 1) found that the data could be fit better with a double power law. In these specifications, with a threshold of $z_0 = 1.105$, the tail parameter, α, was smaller in the part of the distribution with the largest disasters than in the part with the smaller disasters. The cutoff value for the two parts was at a value of z around 1.4.

[7] We also used the approximation $(1+\hat{\rho})(1+g)^{\alpha} \approx 1+\hat{\rho}+\alpha g$.

D. Maturity of the option

Equation (20) applies when the maturity of the put option is one "period." We now take account of the maturity of the option. In continuous time, the parameter p, measured per year, is the Poisson hazard rate for the occurrence of a disaster. Let T, in years, be the maturity of the (European) put option. The density, h, for the number of hits (disasters) over T is given by[8]

$$
\text{(21)} \qquad \begin{aligned} h(0) &= e^{-pT}, \\ h(1) &= pTe^{-pT}, \\ &\cdots \\ h(x) &= \frac{(pT)^x e^{-pT}}{x!}, x = 0,1,\ldots \end{aligned}
$$

If pT is much less than 1, the contribution to the options price from two or more disasters will be second-order, relative to that from one disaster. For given p, this condition requires consideration of maturities, T, that are not "too long." In this range, we can proceed as in our previous analysis to consider just the probability and size of one disaster. Then, in equation (20), p will be replaced as a good approximation by pT.

The discount rate, $\hat{\rho}$, and growth rate, g, in equation (20) will be replaced (approximately) by $\hat{\rho}T$ and gT. For given $\hat{\rho}$ and g, if T is not "too long," we can neglect these discounting and growth terms. The impacts of these terms are of the same order as the effect from two or more disasters, which we have already neglected.

When T is short enough to neglect multiple disasters and the discounting and growth terms,[9] the formula for the relative options price simplifies from equation (20) to

[8]See Hogg and Craig (1965, p. 88).

[9] The possibility of two disasters turns out to introduce into equation (22) the multiplicative term: $1 + \mathrm{pT} \cdot \left\{-1 + 0.5 \cdot \left[\frac{\alpha z_0^{\alpha}[1+2(\alpha-\gamma)+(\alpha-\gamma)(1+\alpha-\gamma)[\log\left(\frac{1}{\varepsilon}\right)-2\log(z_0)]}{(\alpha-\gamma)(1+\alpha-\gamma)}\right]\right\}$, assuming $\frac{1}{\varepsilon} > z_0^2$, so that two disasters just at the threshold size are not sufficient to get the option into the money. The full term inside the large brackets has to be positive, so that this multiplicative term is increasing in T. The effects from the discount rate and growth rate add multiplicative terms that look like $(1 - \mathit{positive\ constant} \cdot rT)$ and $(1\text{-}\mathit{positive\ constant} \cdot gT)$. Hence, these multiplicative terms are decreasing in T. The overall effect of T implied by the combination of the three

(22) $$\Omega = \frac{\alpha z_0^{\alpha} \cdot pT \cdot \varepsilon^{1+\alpha-\gamma}}{(\alpha-\gamma)(1+\alpha-\gamma)} .$$

Here are some properties of the options-pricing formula:

- The formula for Ω, the ratio of the options price to the stock price, is well-defined if $\alpha > \gamma$, the condition noted before that ensures the finiteness of various rates of return.
- The exponent on maturity, T, equals 1.
- The exponent on the relative exercise price, ε, equals $1 + \alpha$-γ, which is constant and greater than 1 because $\alpha > \gamma$. We noted before that α ranged empirically between 5 and 8. The corresponding range for γ (needed to replicate an average unlevered equity premium of 0.04-0.05 per year) is between 2.5 and 5.5, with lower γ associating with lower α. The implied range for α-γ (taking account of the association between γ and α) is between 2.5 and 4.5, implying a range for the exponent on ε between 3.5 and 5.5.
- For given T and ε, Ω depends on the disaster probability, p; the shape of the power-law density, as defined by the tail coefficient, α, and the threshold, z_0; and the coefficient of relative risk aversion, γ. The expression for Ω is proportional to p.
- For given p and γ, Ω rises with a once-and-for-all shift toward larger disaster sizes; that is, with a reduction in the tail coefficient, α, or an increase in the threshold, z_0.
- For given p, α, and z_0, Ω rises for sure with a once-and-for-all increase in γ if $\varepsilon \leq 1$, which is the range that we are considering for put options. Note that, in contrast, the Black-Scholes options-pricing formula implies that Ω is independent of γ.[10]

multiplicative terms is unclear. That is, it is unclear how the full result for Ω would deviate from unit elasticity with respect to T.

[10]See, for example, Hull (2000, pp. 248 ff.). However, this standard result depends on holding fixed the risk-free rate, r^f. Equation (7) shows that r^f depends negatively on γ.

We can look at the results in terms of a measure of "risk-neutral probability," p^n, which we define as the value of p that generates a specified relative options price, Ω, when $\gamma = 0$. This version of "risk-neutral probability" relates to the usual approach, which multiplies p by the marginal utility of consumption in a disaster state relative to that in a normal state. Since our model allows for disasters to differ in severity according to the power-law distribution, there are multiple disaster states. The expression for p^n is calculated assuming that the disaster-size distribution remains as specified under the physical measure. Therefore, we obtain the risk-neutral probability of a disaster occurrence when maintaining the relative frequency of disaster severity that corresponds empirically to the macroeconomic estimates. The formula for the ratio of the risk-neutral to the objective probability, p^n/p, implied by equation (22) is

$$\frac{p^n}{p} = \frac{\alpha(1+\alpha)}{(\alpha-\gamma)(1+\alpha-\gamma)} \cdot \varepsilon^{-\gamma} \ . \qquad (23)$$

Note that p^n/p depends on the relative exercise price, ε, but not the maturity, T. Thus, the compensation for risk differs for options of varying moneyness. If we assume parameter values consistent with the previous discussion—for example, $\alpha = 7$ and $\gamma = 3.5$—the implied p^n/p is 5.1 when $\varepsilon = 0.9$, 7.8 when $\varepsilon = 0.8$, 12.4 when $\varepsilon = 0.7$, 21.3 when $\varepsilon = 0.6$, and 40.3 when $\varepsilon = 0.5$. Hence, the relative risk-neutral probability associated with far-out-of-the-money put options is sharply above one.

To view it another way, the relative options price, Ω, may seem far too high at low ε, when assessed in terms of the (risk-neutral) probability needed to justify this price. Thus, people who are paying these prices to insure against the risk of an enormous disaster may appear to be irrational. In contrast, the people writing these far-out-of-the-money puts may seem to be getting free money by insuring against something that is virtually impossible. Yet the pricing is reasonable if people have roughly constant relative risk aversion with γ of 3-4 (assuming a tail parameter, α, for disaster size around 7). The writers of these options will have a comfortable income almost all the time, but will suffer tremendously during the largest rare disasters, when the marginal utility of consumption is extremely high.

E. Diffusion term

The formula for Ω, the relative options price, in equation (22) neglects the diffusion term, u, in the process for GDP (and consumption and the stock price) in equation (1). This omission is satisfactory if the put option is sufficiently far out of the money so that, given a reasonable variance σ^2 of the diffusion term, the chance that the diffusion shock on its own gets the option into the money over the maturity T is negligible. In other words, the tail for the normal process is not fat enough to account by itself for, say, 10% or greater declines in stock prices over periods of a few months. Operationally, our main empirical analysis applies to options that are at least 10% out of the money ($\varepsilon \leq 0.9$) and to maturities, T, that range up to 6 months.

If we consider put options at or close to the money, the diffusion term would have a first-order impact on the value of the option. If we neglect the disaster (jump) term—which will be satisfactory here—we would be in the standard Black-Scholes world. In this setting (with i.i.d. shocks), a key property of the normal distribution is that the variance of the stock price over interval T is proportional to T, so that the standard deviation is proportional to the square root of T. This property led to the result in Brenner and Subrahmanyam (1988) that the value of an at-the-money put option is proportional to the square root of the maturity.

We, therefore, have two theoretical results concerning the impact of maturity, T, on the relative options price, Ω. For put options far out of the money (operationally for $\varepsilon \leq 0.9$), the exponent on T is close to 1. For put options close to the money (operationally for $\varepsilon = 1$), the exponent on T is close to one-half.

F. Time-varying disaster probability

The asset-pricing formula in equation (22) was derived under the assumption that the disaster probability, p, and the size distribution of disasters (determined by α and z_0) were fixed.[11] However, these assumptions about the disaster process turn out to be counter-factual. In this section, we assume that the disaster probability p is itself governed by a Poisson process. Specifically, unusual and sharp increases in p resemble disasters. This specification turns out to work well empirically. The goal of this section is to enhance the model to incorporate the time-variation in p and, thereby, correct the misspecification error that results from constraining p to be constant.

We focus here on shifting p, but the results are isomorphic to shifting disaster distribution (reflecting changes in α and z_0). The assumptions of a constant relative risk-aversion coefficient, γ, and a constant disaster-size distribution, governed by z_0 and α, allow us identify p_t from the data and interpret p_t as the time varying physical probability of disaster.

We can rewrite equation (22) as

$$\Omega = \eta_1 \mathrm{pT} \varepsilon^{1+\alpha-\gamma}, \tag{24}$$

where $\eta_1 = \frac{\alpha z_0^{\alpha}}{(\alpha-\gamma)(1+\alpha-\gamma)}$ is a constant. We can estimate equation (24) with data on Ω for far-out-of-the-money put options on, say, the S&P 500. Given ranges of maturities, T, and relative exercise prices, ε, we can estimate elasticities of Ω with respect to T and ε. We can also test the hypothesis that $\eta_1 p$ is constant. Using month-end data on put options for several stock-market indices, we estimated monthly fixed effects and tested the hypothesis that these fixed effects were all equal for each stock-market index. The results, detailed in a later section, strongly reject

[11]We also assumed that preference parameters, including the coefficient of relative risk aversion, γ, are fixed.

the hypothesis that $\eta_1 p$ is constant. Instead, the estimated monthly fixed effects fluctuate dramatically, including occasional sharp upward movements followed by gradual reversion over several months toward a small baseline value. From the perspective of the model, if we assume that α, γ, and z_0 are fixed, so that η_1 is constant, these shifts reflect variations in the disaster probability, p.

If $\gamma > 1$, as we assume, equation (8) implies that a once-and-for-all rise in disaster probability, p, lowers the price-dividend ratio, V, if $\theta < 1$ (meaning that the intertemporal elasticity of substitution, $1/\theta$, exceeds 1).[12] Bansal and Yaron (2004) focus on $IES > 1$ because it corresponds to the "normal case" where an increase in the expected growth rate, g^*, raises V. Barro (2009) argues that $IES > 1$ is reasonable empirically and, therefore, also focuses on this case.

Generally, the effects on options pricing depend on θ and other parameters and also on the stochastic process that generates variations in p, including the persistence of these changes. However, for purposes of pricing stock options, we need only consider the volatility of the overall term, $\eta_1 p$, which appears on the right side of equation (24). Our first-round look at the data—that is, the estimated monthly fixed effects—suggested that this term looks like a disaster process. On rare occasions, this term shifts sharply and temporarily upward and leads, thereby, to a jump in the corresponding term in equation (24). We think of this shock as generated by another Poisson probability, q, with a size distribution (for changes in stock prices) involving another power-law distribution, in this case with tail parameter $\alpha^* > \gamma$. If this process for changing p is independent of the disaster realizations (which depend on the level of p), then equation (22) is modified to

[12]Under the same conditions, a fall in α or a rise in σ^2 reduces V.

(25) $$\Omega = \frac{\alpha z_0^{\alpha} \cdot p_t T \cdot \varepsilon^{1+\alpha-\gamma}}{(\alpha-\gamma)(1+\alpha-\gamma)} + \frac{\alpha*(z_0^{*})^{\alpha*} \cdot qT \cdot \varepsilon^{1+\alpha*-\gamma}}{(\alpha*-\gamma)(1+\alpha*-\gamma)}.$$

The first term on the right side of equation (25) reflects put-option value associated with the potential for realized disasters, and the second term gauges value associated with changing p_t and the effects of these changes on stock prices.[13] The logic of the second term differs from the first term, in that the first term reflects the fall in consumption in a disaster whereas the second term reflects changes in the distribution of future payoffs. The inclusion of p_t in the first term is an approximation that neglects the tendency for p_t to revert over time toward a small baseline value. This approximation for options with relatively short maturity is similar to others already made, such as the neglect of multiple disasters and the ignoring of discounting and expected growth.

We can rewrite the formula in equation (25) as

(26) $$\Omega = \mathrm{T}\varepsilon^{1+\alpha-\gamma} \cdot [\eta_1 p_t + \eta_2 q \varepsilon^{(\alpha^*-\alpha)}],$$

where

(27) $$\eta_1 = \frac{\alpha z_0^{\alpha}}{(\alpha-\gamma)(1+\alpha-\gamma)}, \eta_2 = \frac{\alpha^*(z_0^*)^{\alpha^*}}{(\alpha^*-\gamma)(1+\alpha^*-\gamma)}$$

are constants.[14] The new term involving $\eta_2 > 0$ turns out to be important for fitting the data on put-options prices. Notably, this term implies $\Omega > 0$ if $p_t = 0$ because of the possibility that p_t will rise during the life of the option. The preclusion of changing p_t (corresponding to $\eta_2 = 0$) leads, as emphasized by Seo and Wachter (2016), to overestimation of the average level of p_t in the sample.[15] In addition, our hypotheses about elasticities of Ω with respect to T and ε in equation (26) turn out to accord better with the data when $\eta_2 > 0$ is admitted.

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[13]The formulation would also encompass effects on stock prices from changing α or γ.

[14]These values are constant if α, α^*, z_0, $(z_0)^*$, γ, and q are all constant. $\alpha^* - \alpha$ is identified in our estimation because we have sample variation in relative exercise prices, ε.

[15]As Seo and Wachter (2016) note, these problems appear, for example, in Backus, Chernov, and Martin (2011).

IV. Empirical Analysis

The model summarized by equation (26) delivers testable predictions. First, the elasticity of the put-options price with respect to maturity, T—denoted β_T—is close to one. Second, for a given value of $\eta_2 q \varepsilon^{(\alpha^* - \alpha)}$, the elasticity of the put-options price with respect to the relative exercise price, ε—denoted β_ε—is greater than one and corresponds to $1 + \alpha - \gamma$. Given a value of γ and the estimated value of $\alpha^* - \alpha$ from equation (26), the results can be used to back out estimates of the tail parameters α and α^*. Finally, the monthly fixed effects provide estimates of each period's disaster probability, p_t (or, more precisely, of p_t multiplied by the positive constant η_1). We assess these hypotheses empirically by analyzing prices of far-out-of-the-money put options on the U.S. S&P 500 and analogous broad stock-market indices for other countries.

Following the estimation of the model and hypothesis testing, we discuss applications of the estimated p_t to forecasting economic growth. This section concludes with robustness tests, including estimations using alternative data sources.

A. Data

Our analysis relies on two types of data—indicative prices on over-the-counter (OTC) contracts offered to clients by a large financial firm and U.S. market data provided by Berkeley Options Data Base and OptionMetrics. The Berkeley data allow us to extend the U.S. analysis back to 1983, thereby bringing out the key role of the stock-market crash of October 1987. The OptionMetrics information allows us to check whether the results using U.S. OTC data differ from those using market data. We find that the main results are similar with the two types of data.

Our primary data source is a broker-dealer with a sizable market-making operation in global equities. We utilize over-the-counter (OTC) options prices for seven equity-market indices—S&P 500 (U.S.), FTSE (U.K.), DAX (Germany), ESTX50 (Eurozone), Nikkei (Japan), OMX (Sweden), and SMI (Switzerland). The OTC data derive from implied-volatility surfaces generated by the broker-dealer for the purpose of analysis, pricing, and marking-to-market.[16] These surfaces are constructed from transaction prices of options and OTC derivative contracts.[17] The dealer interpolates these observed values to obtain implied volatilities for strikes ranging from 50% to 150% of spot and for a range of maturities from 15 days to 2 years and more. Even at very low strikes, for which the associated options seldom trade, the estimated implied volatilities need to be accurate for the correct pricing of OTC derivatives such as variance swaps and structured retail products. Institutional-specific factors are unlikely to influence pricing in a significant way because other market participants can profitably pick off pricing discrepancies among dealers. Therefore, dealers have strong incentives to maintain the accuracy of their implied-volatility surfaces.

Using the data on implied volatilities, we re-construct options prices from the standard Black-Scholes formula, assuming a zero-discount rate and no dividend payouts. The adjustment in option prices due to zero discount rate and dividend is not large in practice and allows a closer comparison of the model and the data. We should emphasize that this use of the Black-Scholes *formula* to translate implied volatilities into options prices does not bind us to the Black-Scholes *model* of options prices. The formula is used only to convert the available data expressed as

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[16]A common practice in OTC trading is for executable quotes to be given in terms of implied volatility instead of the price of an option. Once the implied volatility is set, the options price is determined from the Black-Scholes formula based on the observable price of the underlying security. Since the Black-Scholes formula provides a one-to-one mapping between price and implied volatility, quotes can be given equivalently in terms of implied volatility or price.

[17]Dealers observe prices through own trades and from indications by inter-dealer brokers. It is also a common practice for dealers to ask clients how their prices compare to other market makers in OTC transactions.

implied volatilities into options prices. Our calculated options prices are comparable to directly quoted prices (subject to approximations related to discounting and dividend payouts).

We sample the data at a monthly frequency, selecting only month-end dates, to allow for ease of computation with a non-linear solver. The selection of mid-month dates yields similar results. The sample period for the United States in our main analysis is August 1994-June 2018. Because of lesser data availability, the samples for the other stock-market indices are shorter. Subsequently, we expand the U.S. sample back to 1983, particularly to assess pricing behavior before and after the global stock-market crash of 1987. However, we do not use this longer sample in our main analysis because the data quality before 1994 is substantially poorer.

The OTC data source is superior to market-based alternatives in the breadth of coverage for exercise prices and maturities. Notably, the market data tend to be less available for options that are far out of the money and for long maturities. The broad range of strikes in the broker-dealer data is important for our analysis because it is the prices of far-out-of-the-money put options that will mainly reflect disaster risk. In practice, we use put options with exercise prices of 50%, 60%, 70%, 80%, and 90% of spot; that is, we exclude options within 10% of spot. For maturities, we focus on the range of 30 days, 60 days, 90 days, and 180 days.[18]

Our main analysis excludes options with maturities greater than six months because the prices in this range may be influenced significantly by the possibility of multiple disaster realizations and also by discounting and expected growth. However, in practice, the results for one-year maturity accord reasonably well with those for shorter maturities.

[18]We omit 15-day options because we think measurement error is particularly serious in this region in pinning down the precise maturity. Even the VIX index, which measures short-dated implied volatility, does not track options with maturity less than 23 days.

Our extended analysis to the period June 1983 to July 1994 uses market-based quotes on S&P 100 index options from the Berkeley Options Data Base.[19] These data derive from CBOE's Market Data Retrieval tapes. Because of the limited number of quotes on out-of-the-money options in this database, we form our monthly panel by aggregating quotes from the last five trading days of each month. The available Berkeley data allow us to consider relative exercise prices, ε, around 0.9, with maturities, T, close to 30, 60, and 90 days. We also have a small amount of data with ε around 0.8 and maturity, T, of about 30 days.

B. Estimation of the Model

We estimate the model based on equation (26) with non-linear least-squares regression. In this form, we think of the error term as additive with constant variance (although we calculate standard errors of estimated coefficients by allowing for serial correlation in the error terms). Log-linearization with a constant-variance error term (that is, a shock proportional to price) is problematic for low-strike options because it understates the typical error in extremely far-out-of-the-money put prices, which are close to zero. That is, this specification would give undue weight to puts with extremely low exercise prices.

In the non-linear regression, we allow for monthly fixed effects to capture the unobserved time-varying probability of disaster, p_t, in equation (26). We allow the estimated p_t to differ across the seven stock-market indices; that is, we estimate index-time fixed effects. Note that, for a given stock-market index and date, these effects are the same for each maturity, T, and relative exercise price, ε. We carry out the estimation under the constraint that all of the index-time fixed effects are non-negative—corresponding to the constraint that all p_t are non-negative. On average for the seven stock-market indices, the constraint of non-negative monthly fixed

[19]Direct access to this database has been discontinued. We thank Josh Coval for sharing his version of the data. We have the data from Berkeley Options Data Base through December 1995.

effects is binding for 8% of the observations. Only a negligible number of the unconstrained estimates of the fixed effects are significantly negative.

Table 1 shows the estimated equations. The regressions apply to each of the seven stock-market indices individually and also to joint estimation with pooling of all of the data. In the last case, we constrain the estimated coefficients (including the monthly fixed effects) to be the same for each stock-market index.

1. Maturity Elasticities. The estimated elasticities with respect to maturity, β_T, are close to one, as hypothesized. For example, the estimated coefficient for the U.S. S&P 500 is 0.992 (s.e. = 0.040) and that for all seven indices jointly is 0.961 (0.042). The only case in which the estimated coefficient differs significantly from 1 at the 5% level is Japan (NKY), where the estimated coefficient is 0.881 (s.e. = 0.032). The p-value for this estimated coefficient to be statistically different from 1 is 0.014. In the main regression table, with the exception of Japan, the results indicate that prices of far-out-of-the-money put options on broad market indices are roughly proportional to maturity, in accordance with the rare-disasters model. This nearly proportional relationship between options price and maturity for far-out-of-the-money put options is a newly documented fact that cannot be explained under the Black-Scholes model.

The unit elasticity of options price with respect to maturity for far-out-of-the-money put options contrasts with the previously mentioned result from Brenner and Subrahmanyam (1988) that prices of at-the-money put options in the Black-Scholes model are proportional to the square root of maturity.[20] This result arises because, with a diffusion process driven by i.i.d. normal shocks, the variance of the log of the stock price is proportional to time and, therefore, the

[20]We verified empirically that the maturity elasticity is close to one-half for at-the-money put options. For the pooled sample with data from all seven stock-market indices, the estimated β_T is 0.499 (s.e. = 0.031). Similar point estimates apply for each of the seven stock-market indices considered individually.

standard deviation is proportional to the square root of time. In contrast, as discussed earlier, the roughly proportional relationship between far-out-of-the-money put prices and maturity arises because, in a Poisson context, the probability of a disaster is proportional to maturity.[21]

2. Elasticities with respect to exercise price. Table 1 shows estimates of the elasticity, β_ϵ, of the put-options price with respect to the relative exercise price (holding fixed the term that involves $\eta_2 q$ in equation [26]). The coefficient β_ϵ corresponds in the model to $1+\alpha-\gamma$, where α is the tail coefficient for disaster sizes and γ is the coefficient of relative risk aversion. The estimated β_ϵ for the various stock-market indices are all positive and greater than one, as predicted by the model. The estimated coefficients are similar across indices, falling into a range from 4.01 to 4.75. The joint estimate across the seven indices is 4.55 (s.e. = 0.45).

Rare-disasters research with macroeconomic data, such as Barro and Ursúa (2008) and Barro and Jin (2011), suggested that a γ of 3-4 would accord with the observed average (unlevered) equity premia. With this range for γ, the estimated values of β_ε in Table 1 suggest that α would be between 6 and 8. This finding compares with an estimate for α based on macroeconomic data on consumption in Barro and Jin (2011, Table 1) of 6.3 (s.e. = 0.8). Hence, the estimates of α implied by Table 1 accord roughly with those found from direct observation of the size distribution of macroeconomic disasters (based on consumption or GDP).

3. Estimated disaster probabilities. We use the estimated monthly fixed effects for each stock-market index from the regressions in Table 1 to construct time series of estimated (objective) disaster probabilities, p_t. Note from equations (26) and (27) that the estimation

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[21]The resulting pricing formula is only approximate because it neglects, for example, the potential for multiple disasters within the time frame of an option's maturity, omits a diffusion term, and also neglects the tendency of the disaster probability to revert over the life of an option toward a small baseline value. However, for options that are not "too long," these approximations will be reasonably accurate, consistent with the findings on maturity elasticity shown in Table 1.

identifies p_t multiplied by the parameter $\eta_1 = \frac{\alpha z_0^\alpha}{(\alpha-\gamma)(1+\alpha-\gamma)}$, which will be constant if the size distribution of disasters (determined by α and z_0) and the coefficient of relative risk aversion, γ, are fixed. When η_1 is constant, the estimated p_t for each stock-market index will be proportional to the estimated monthly fixed effect. The constancy of η_1 is needed to interpret the fluctuations in the estimated p_t as changes in physical probability rather than changes in either risk-aversion or disaster distribution.

To estimate the level of p_t, we need a value for η_1, which depends in equation (27) on α, z_0, and γ. We assume for a rough calibration that the threshold for disaster sizes is fixed at $z_0 = 1.1$ (as in Barro and Jin [2011]) and that the coefficient of relative risk aversion is $\gamma = 3$. We allow the tail coefficient, α, to differ for each stock-market index; that is, we allow prices to differ with respect to the size distribution of potential disasters. We use the estimated coefficients from Table 1 for β_ε (which equals $1 + \alpha - \gamma$ in the model) to back out the implied α, also shown in Table 1. These values range from 6.0 to 6.8, implying a range for η_1 from 0.72 to 0.88. Dividing the estimated monthly fixed effects for each stock-market index by the associated η_1 generates an estimated time series of p_t for each country or region.

These values of p_t are shown for the seven stock-market indices in Figure 1, Panel A. Panel B presents the results just for the United States (SPX), with the standard volatility indicator (VIX) included as a comparison.[22] Panel C shows the results for all indices estimated jointly (last column of Table 1). Note that our assumed parameter values, embedded in the computation of η_1, influence the levels of the p_t series in Figure 1, but not the time patterns.

[22]A discussion of the VIX is contained in Chicago Board Options Exchange (2014).

Table 2 provides summary statistics for the estimated disaster probabilities. These probabilities, shown in Figure 1, Panel A, have high correlations among the countries, with an average pair-wise correlation for the monthly data of 0.89. The high correlations across stock-market indices suggest that the main variations in inferred disaster probabilities reflect the changing likelihood of a common, global disaster.

The mean estimated disaster probability from Table 2 is 6.2% per year for the S&P 500 and 6.1% for all countries jointly. For the other indices, the means range from 4.6% for Japan (NKY) to 9.0% for Sweden (OMX). These estimates can be compared with average disaster probabilities of 3-4% per year estimated from macroeconomic data on rare disasters—see, for example, Barro and Ursúa [2008]). However, this earlier analysis assumed that disaster probabilities were constant across countries and over time.

The estimated disaster probabilities in Figure 1 are volatile and right-skewed, with spikes during crisis periods and lower bounds close to zero. The U.S. disaster probability hit a peak of 42% per year in October-November 2008, just after the Lehman crisis. Similarly, the other six stock-market indices show their highest disaster probabilities around 40% in October-November 2008. Additional peaks in disaster probability occurred around the time of the Russian and Long-Term Capital Management (LTCM) crises in August-September 1998. In this case, the estimated U.S. disaster probability reached 29% in August 1998.

The patterns found for the U.S. disaster probability mirror results for options-derived equity premia in Martin (2015) and for disaster probabilities in Siriwardane (2015). The U.S. disaster probability is also highly correlated with the Chicago Board Options Exchange's well-known volatility index (VIX), as indicated in Figure 1, Panel B. The VIX, based on the S&P 500 index, is from a weighted average of puts and calls with maturity between 23 and 37 days and for an array of exercise prices. The correlation between the VIX and our U.S. disaster probability (using month-end data from August 1994 to June 2018) is 0.96 in levels and 0.90 in monthly changes. However, the levels of the two series are very different, with the p_t series having the interpretation as an objective disaster probability (per year) and the VIX representing the fair strike of a variance swap contract (with units of annualized standard deviation of stock-price changes). Additionally, the p_t series and the VIX have differentiated power in forecasting equity returns as discussed in a later section.

How unlikely is the no-disaster world that we have apparently experienced in the sample period given the market-implied disaster probabilities? The cumulative survival probability—the probability of not having experienced a disaster—from 1994 to 2018 for the United States considered in isolation is 22%. The range for other countries considered individually is similar, from 20% (Sweden) to 39% (Japan) during the relevant sample period for each country. A substantial amount of disaster risk was priced by the options market around the 2008-09 financial crisis. During the period 2008-2010, the cumulative probabilities of experiencing at least one disaster were 36% for the United States and of similar magnitude for the other places.

The estimated U.S. disaster probability, p_t , is positively but not that strongly correlated with the indexes of economic policy uncertainty (EPU) constructed by Baker, Bloom, and Davis (2016). From August 1994 to June 2018, the correlation of our p_t series with their news-based uncertainty measure was 0.41 and that with their broader uncertainty measure was 0.46. As an example of a deviation, in January-February 2017, our p_t series was around 1% (compared to a mean of 6.2%), the news-based policy uncertainty indicator was close to 200 (mean of 113), and the broader policy uncertainty indicator was about 140 (mean of 106). In other words, disaster probability was low (according to the financial markets), while policy uncertainty was high (according to Baker, Bloom, and Davis [2016] and, presumably, most political commentators).

The estimated first-order AR(1) coefficient for the estimated U.S. disaster probability is 0.88 (s.e. = 0.03), applying at a monthly frequency. This coefficient implies that shocks to disaster probability have a half-life around eight months. The persistence of disaster probabilities for the other stock-market indices (Figure 1) is similar to that for the United States, with estimated AR(1) coefficients ranging from 0.85 to 0.89, except for Japan at 0.80. An important inference is that the movements in disaster probability shown in Figure 1 are temporary. The series is associated with occasional sharp upward spikes (involving the probability q), followed by reasonably quick reversion toward a small baseline value.

Although we attributed the time pattern in Figure 1 to variable disaster probability, p_t, the variations in the monthly fixed effects may also reflect changes in the other parameters contained in the term that multiplies p_t in equation (26) and is shown in equation (27) as $\eta_1 = \frac{\alpha z_0^\alpha}{(\alpha-\gamma)(1+\alpha-\gamma)}$.[23] For example, outward shifts in the size distribution of disasters, generated by reductions in the tail parameter, α, or increases in the threshold disaster size, z_0, would work like increases in p.[24] Similarly, increases in the coefficient of relative risk aversion, γ, would raise η_1. This kind of change in risk preference, possibly due to habit formation, has been stressed by Campbell and Cochrane (1999). Separation of changes in the parameters of the disaster distribution from those in risk aversion requires simultaneous consideration of asset-

[23]In the model with i.i.d. shocks, this term does not depend on the intertemporal elasticity of substitution for consumption, $1/\theta$, or the rate of time preference, ρ.
[24]Kelly and Jiang (2014, p. 2842) assume a power-law density for returns on individual securities. Their power law depends on a cross-sectional parameter and also on aggregate parameters that shift over time. In contrast to our analysis, they assume time variation in the economy-wide values of the tail parameter, analogous to our α, and the threshold, analogous to our z_0. (Their threshold corresponds to the fifth percentile of observed monthly returns.)

pricing effects (reflected in Figure 1) with information on the incidence and sizes of disasters (based, for example, on movements of macroeconomic variables).

4. Coefficients associated with changing disaster probability. The options-pricing formula in equation (26) involves the probability, q, of an upward jump in disaster probability, p_t. The probability q enters multiplicatively with η_2, given in equation (27). In other words, $\eta_2 q$ is identified in the data.[25] The results in Table 1 show that the estimates of $\eta_2 q$ range from 0.078 to 0.102, except for Japan at 0.128. Note that the underlying values of $\eta_2 q$ are assumed to be constant over time for each stock-market index; we consider later whether this restriction is satisfactory. The effect of $\eta_2 q$ on the options price interacts in equation (26) with the exercise price, ε, to the power α^*-α. Because the sample for each stock-market index has variation each month in ε, the non-linear estimation identifies α^*-α. These estimates range, as shown in Table 1, from 7.9 to 10.9.

Note in equation (26) that the put-options price, Ω, depends on the sum of $\eta_1 p_t$ and $\eta_2 q$, with the second term multiplied by $\varepsilon^{\alpha^*-\alpha}$. Given that α^*-α is estimated to be around 9.5, this last term ranges from 0.001 when $\varepsilon = 0.5$ to 0.45 when $\varepsilon = 0.9$. Options pricing depends, accordingly, on an effective probability that weighs the current disaster probability, p_t, along with the probability, q, of a sharp upward future rise in p_t. From this perspective, it is clear that omitting the chance of future rises in disaster probability—that is, assuming $q = 0$—will result in estimates of p_t that are too high on average compared with objective probabilities of disasters. Moreover, this effect will be much more significant at high exercise prices, such as $\varepsilon = 0.9$, than

[25]We can therefore identify q if we know the value of $\eta_2 = \frac{\alpha^*(z_0^*)^{\alpha^*}}{(\alpha^*-\gamma)(1+\alpha^*-\gamma)}$. If we continue to assume $\gamma = 3$ and use the estimates of α^* implied by the results in Table 1, the missing element is the threshold, z_0^*. However, reasonable variations in z_0^* imply large variations in η_2 and, hence, in the estimated q.

at low ones, such as $\varepsilon = 0.5$. For very low exercise prices, such as $\varepsilon = 0.5$, almost all of the option value reflects the chance of a realization of a disaster during the life of the option. In contrast, for high exercise prices, such as $\varepsilon = 0.9$, the option value depends partly on the possibility of a disaster occurrence and partly on the possibility of p_t rising sharply.

As noted before, the term involving $q > 0$ in equation (26) implies $\Omega > 0$ even when $p_t = 0$. For example, using the estimated value $\eta_2 q = 0.10$ (from the pooled sample in Table 1) and taking $p_t = 0$, the term $\eta_1 p_t + \eta_2 q \varepsilon^{\alpha^* - \alpha}$ in equation (26) equals 0.037 when $\varepsilon = 0.9$. That is, the "effective probability" that determines Ω can be as high as 4% per year even though $p_t = 0$ applies.

5. Long-term results for the United States. A lot of analysis of options pricing, starting with Bates (1991), suggests that the nature of pricing changed in character following the October 1987 stock-market crash. In particular, a "smile" in graphs of implied volatility against exercise price is thought to apply only post-1987. As noted before, we expanded our analysis to the period June 1983 to July 1994 by using market-based quotes from the Berkeley Options Data Base.

Table 3 extends the analysis of put-options pricing from Table 1 to consider U.S. regression estimates over the longer period 1983-2018. In this estimation, the data from Berkeley Options Data Base (June 1983 to July 1994) relate to the S&P 100 but are treated as comparable to the OTC data (August 1994-June 2018) associated with the S&P 500. The estimates of the various coefficients are close to those shown in Table 1, which were based on data from August 1994 to June 2018.

As before, we back out a time series for estimated disaster probability, p_t, based on the monthly fixed effects, assuming that the parameters in the term η_1 in equations (26) and (27) that involve p_t are fixed. We use levels for these other parameters similar to those used before (including $\eta_1 = 0.73$). Figure 2 graphs the resulting time series of estimated U.S. disaster probability. Readily apparent is the dramatic jump in p_t at the time of the October 1987 stock-market crash, in which the S&P 500 declined by 20.5% in a single day. The estimated p_t reached 135% per year but fell rapidly thereafter.[26] The Persian Gulf War of 1990-1991 caused another rise in disaster probability, to 19-20%.

The bottom part of Table 3 shows statistics associated with the time series in Figure 2. A comparison pre-crash (June 1983-Sept 1987) and post-crash (Oct 1988-July 1994), based on the data from the Berkeley Options Data Base, shows an increase in the typical size and volatility of the estimated disaster probability, p_t. The change in mean is from 0.004 to 0.029, and the change in standard deviation is from 0.007 to 0.047. The period August 1994-June 2018, based on OTC data related to the S&P 500, shows further rises in mean and standard deviation—to 0.070 and 0.071, respectively. Thus, the overall suggestion is that the mean and standard deviation of the disaster probability shifted permanently upward because of the October 1987 stock-market crash.

C. Predictive Power of Disaster Probability for Economic Growth

We use our estimated disaster probability to forecast the conditional distribution of economic growth. Since the disaster probability applies to left-tail events, we use p_t to forecast lower quantiles of growth in U.S. annual and quarterly GDP and monthly industrial production (IP). The approach follows Adrian, Boyarchenko, and Giannone (2019), who

[26] A probability or hazard rate above one at a point in time is consistent with the realized disaster probability being below one in any finite period. For example, a probability of 135% per year implies a disaster probability of 89% over one month.

examine the conditional distribution of GDP growth as a function of the Financial Condition Index.

We estimate quantile forecasting regressions of one-year and one-quarter ahead U.S. real GDP growth and one-month ahead IP growth on lagged disaster probabilities and lagged GDP growth at matching horizons. Specifically, the ordinary least squares forecasting regression is specified as $\Delta GDP_{t\to t+h} = \beta_0 + \beta_1 \Delta GDP_{(t-h)\to t} + \beta_1 p_t + \epsilon,$ where $\Delta GDP_{t\to t+h} = \frac{GDP_{t+h}-GDP_t}{GDP_t}$ and p_t is the last monthly value observed before t. One-year GDP growth is measured as the year-over-year proportionate change in real GDP observed at a quarterly frequency. One-quarter GDP growth is measured as the quarter-over-quarter seasonally-adjusted annual rate. The quantile forecasting regressions estimate coefficients at specified quantile τ by minimizing the weighted absolute value of residuals:

$$\hat{\beta}(\tau) = argmin_{\beta_\tau} \sum_{t=1}^{T-h} \left(\tau - \mathbf{1}_{\{Y_{t+h}-X_t'\beta_\tau \le 0\}}\right) (Y_{t+h} - X_t'\beta_\tau),$$

where h is the forecasting horizon and $\mathbf{1}$ is an indicator function that equals one if the residual is negative and zero otherwise. The predicted value from this regression is the τ-quantile of Y_{t+h} given X_t, $Q(Y_{t+h}|X_t) = X_t'\beta_\tau$.

Figure 3 presents the coefficients of the quantile regressions along with the OLS estimates. The left side uses the estimated disaster probability, p_t, and the right side uses the economic policy uncertainty index (EPU). The results suggest that the information content of p_t(based on data from financial instruments) is distinct from that of policy uncertainty (derived from media reports).

The left panels of Figure 3 show that the slope of growth in response to disaster probabilities varies across quantiles. At the lower growth quantiles, the slopes of growth in response to lagged disaster probabilities, observed before the growth measurement period, are distinct from the OLS estimates at the 5% significance level constructed using Newey-West adjusted standard errors. A one percentage point increase in disaster probability is associated with subsequent declines in GDP (one-quarter and one-year decline of 0.21% per year) and IP (one-month decline of 0.33% per year) for the lowest decile of the growth distribution. This vulnerability to disaster risk is diminished for the upper quantiles of growth. The OLS coefficient estimates are statistically insignificant. This result suggests that p_t is a poor predictor of economic growth on average but is more informative about the left-tail growth outcomes. As a comparison, the right panels show that an increase in the EPU index is not significantly associated with lower future growth. Moreover, the growth response to EPU is symmetric across quantiles, suggesting that EPU does not relate particularly to disaster risk.

The estimated objective disaster probability can also be applied in other contexts. Recent work by Chodorow-Reich, Karabarbounis, and Kekre (2019) applies our model to estimate disaster probability in the Greek economy using options prices traded on the Athens Stock Exchange from 2001 and 2017. The authors find that the elasticity of the options price with respect to moneyness and time are similar to our estimates for other countries. The peak of disaster probability coincides with major political and economic events during the crisis period. The time series is then used to calibrate a macro model.

D. Results with OptionMetrics data on put-options prices

One possible shortcoming of the results in Table 1 is that they are based on underlying OTC data that represent menus of options prices offered to clients by a large financial firm. Although these menus are informed by market transactions, they do not necessarily correspond to actual trades.

To check whether the reliance on OTC data is an issue, we redid the U.S. analysis shown in Table 1 using market-based information from OptionMetrics on far-out-of-the-money put options based on the S&P 500 index. As in Table 1, these data cover options with relative exercise prices, ε, of 0.5, 0.6, 0.7, 0.8, and 0.9, and maturities of 30, 60, 90, and 180 days. The sample is from January 1996 to December 2017. Unfortunately, we lack comparable data for other countries.[27] The regression results with the OptionMetrics data are in Table 4.

The number of observations for the OptionMetrics sample in Table 4 is 3886, compared to 5740 for the U.S. SPX in Table 1. The main reason for the decline in sample size is missing data from OptionMetrics, not the truncation of the sampling interval. Despite the reduction in sample size, it is clear that the OptionMetrics data provide a great deal of coverage over a long period on far-out-of-the-money put options on the S&P 500.

The main inference from Table 4 is that the estimated coefficients and fit using OptionMetrics data are close to those based on the U.S. OTC data in Table 1. One likely reason for this correspondence is that the producers of the OTC information take account of market data, including those that appear in OptionMetrics. In any event, the closeness in results for OTC and market data for the United States makes us more comfortable with the OTC results for the other six stock-market indices, for which we lack long-term market-based information on put-options prices.

E. Test of model robustness

In the underlying theory, the asset-pricing formula in equation (26) applies as an approximation—based, for example, on neglecting possibilities of multiple disasters, neglecting pricing implications of a diffusion term, and ignoring effects from the tendency of p_t to revert

[27] We have data from Bloomberg but only since late 2010.

over time toward a small baseline value. More generally, properties such as $\beta_T = 1$ (and constant) and $\beta_\varepsilon = 1 + \alpha - \gamma$ (and constant) would not hold precisely. In this section, we explore the empirical robustness of the model estimated in Table 1 under various scenarios.

1. Constancy of the maturity elasticity, β_T. We re-estimated the regressions in Table 1 while allowing for different values of β_T over ranges of maturity, T. As an example, we estimated one value of β_T for T equal to 30 or 60 days and another for T equal to 90 or 180 days. For the United States (SPX), the estimated β_T is 0.985 (s.e. = 0.036) in the low range of T and 0.946 (0.052) in the high range, with a p-value of 0.33 for equality of these two coefficients. Similarly, for all stock-market indices estimated jointly, the estimated β_T is 0.954 (s.e. = 0.028) in the low range of T and 0.911 (0.055) in the high range, with a p-value of 0.33 for equality of these two coefficients.

We also redid the regressions in Table 1 while expanding the sample to include put options with one-year maturity. For the United States (SPX), the estimated β_T becomes 0.946 (s.e. = 0.038), compared to 0.992 (0.040) in Table 1, which allows for maturities only up to six months. For all stock-market indices estimated jointly, the estimated β_T with the inclusion of one-year maturity becomes 0.907 (s.e. = 0.039), compared to 0.961 (0.031) in Table 1.

The general pattern is that the estimated β_T declines with the inclusion of longer maturities. However, the effects are moderate even for a range of T up to one year. These findings support the underlying approximations in the model but also suggest that the sample should be restricted to options that are not overly long; for example, up to six months.

We also considered whether β_T is the same over different ranges of exercise price, ε (knowing that, for $\varepsilon = 1$—at-the-money options—β_T would be close to 0.5). We re-estimated the regressions in Table 1 while allowing for different values of β_T over various ranges of ε. As an example, we estimated one value of β_T for ε equal to 0.5, 0.6, or 0.7 and another for ε equal to 0.8 or 0.9. For the United States (SPX), the estimated β_T is 1.201 (s.e. = 0.125) in the low range of ε and 0.973 (0.039) in the high range, with a p-value of 0.070 for equality of these two coefficients. Similarly, for all stock-market indices estimated jointly, the estimated β_T is 1.189 (s.e. = 0.085) in the low range of ε and 0.940 (0.040) in the high range, with a p-value of 0.001 for equality of these two coefficients. Thus, there is some indication that β_T is lower at high ε than at low ε. However, even for ε as high as 0.9, the estimated β_T remains close to 1.

2. Stability of coefficients associated with exercise price. We also checked whether the coefficient β_ε in Table 1 is stable over various ranges of ε. As an example, we estimated one β_ε for ε equal to 0.5, 0.6, or 0.7 and another for ε equal to 0.8 or 0.9. For the United States (SPX), the estimated β_ε is 4.52 (s.e. = 0.27) in the low range of ε and 4.42 (0.88) in the high range, with a p-value of 0.45 for a test of the equality of these coefficients. Similarly, for all stock-market indices estimated jointly, the estimated β_ε is 4.16 (s.e. = 0.27) in the low range of ε and 4.27 (0.84) in the high range, with a p-value of 0.48 for a test of the equality of these coefficients. Thus, these results are consistent with the stability of the coefficient β_ε over ranges of ε.

3. Maturity-varying epsilon threshold. We test our model with an alternative sampling of option prices that are maturity-dependent. This is because an epsilon of 0.9 could be considered far out-of-the-money at the one-month horizon, but this threshold might not be enough at longer maturities. We vary the upper epsilon threshold by the maturity of the option in two ways. First, we modify the threshold to 0.8 for options greater than one month but keep options with an epsilon of 0.9 for one-month options. Second, we expand the sample of option prices to include options with epsilon at every 0.025 interval (through interpolating the implied volatility surface and reconstructing the more granular options prices). We then calibrate a different cutoff at different maturities based on a Brownian diffusion process in which the options are rarely in the money based on diffusion alone. Table 5 presents the findings of this exercise. We find that the regression coefficients largely accord with our main model. The estimates are β_{ε} = 5.10 (s.e. = 0.58) and β_{T} = 1.12 (s.e. = 0.083) using the first method described above. The second method yields similar estimates.

4. Different sample periods. We checked for the stability of the regression coefficients over time by re-estimating the regressions in Table 1 with separate coefficients for the four sub-periods shown in Table 6. These periods are Aug 1994–Jan 2003, Feb 2003-Feb 2008, Mar 2008-Mar 2013, and Apr 2013-Jun 2018. These intervals were chosen to be of roughly equal length, starting from January 1998, at which point five of the seven stock-market indices have data. The results in Table 6 are for the United States (SPX) and for the pooled sample with data for the seven indices.

The general pattern in Table 6 is that the estimated coefficients are reasonably stable across the sub-periods, although hypotheses of equality of coefficients over time tend to be rejected at usual critical levels. For example, for the maturity elasticity, β_{T}, the range of estimated values over the four sub-periods is fairly narrow for the U.S. data—0.998 (s.e. = 0.043), 1.203 (0.048), 0.920 (0.042), and 1.303 (0.057). Similar results obtain for the pooled sample of seven indices. Despite the narrow range of estimates, the hypothesis of equality is rejected in each case with a p-value of 0.000 because the estimated coefficients have high precision.

Similarly, for the coefficient β_{ε} related to exercise-price elasticity, the range of estimates for the United States is from 4.00 to 6.30, and the hypothesis of equal coefficients is rejected with a p-value of 0.000. Analogous findings apply for the sample of seven stock-market indices.

For the estimated value of $\alpha^* - \alpha$, the range is wider—from 4.92 to 10.65 for the United States. The p-value for the hypothesis of equal coefficients has a higher p-value, 0.055.

Finally, the estimated value of $\eta_2 q$ for the United States is from 0.072 to 0.098, and the p-value for equal coefficients is 0.084. For the seven-index sample, the range is from 0.077 to 0.111. These results support the assumption that, at least since August 1994, the probability, q, of a sharp upward movement in disaster probability, p_t, is relatively stable. That is, unlike the dramatic variations in p_t itself, it seems reasonable to assume time invariance with regard to the volatility associated with potential variations in p_t.

5. Estimated Diffusion Term. Our estimated pricing formula for far-out-of-the-money options, where $\epsilon \leq 0.9$, can be used to estimate the pricing effects from the usual diffusion term for near-the-money options, where $0.9 < \epsilon \leq 1$. We start by using the regression results for the overall sample of countries in Table 1 to calculate fitted values of Ω for $0.9 < \epsilon \leq 1$ and a specified value of T. We assume that these fitted values reflect the disaster component of options prices. We then assume that the observed values of Ω in the near-the-money range also include a significant diffusion component. Therefore, the difference between the observed and fitted values of Ω gives our estimate of the diffusion component.

Figure 5 gives the results for short maturity, where $T = 1$ month (0.083 years). The results shown are for averages associated with the S&P 500 from 1994 to 2018. The horizontal axis has values of ε from 0.9 to 1.0, and the vertical axis shows averages for Ω and its estimated breakdown into disaster-risk and diffusion components. The disaster-risk component represents nearly the entirety of put prices for options with ϵ not far above 0.9. However, the disaster-risk share falls as ε rises toward 1.0, at which point the disaster-risk component is 59% of the total. Our estimated decomposition of options prices into disaster-risk and diffusion components is analogous to results in Bollerslev and Todorov (2011), who find that around three-fourths of the variance risk premium is attributable in short-maturity options to large tail risks.

V. Conclusions

Options prices contain rich information on market perceptions of rare disaster risks. We develop a new options-pricing formula that applies when disaster risk is the dominant force, the size distribution of disasters follows a power law, and the economy has a representative agent with Epstein-Zin utility. The formula is simple but its main implications about maturity and exercise price accord with U.S. and other data from 1983 to 2018 on far-out-of-the-money put options on broad stock-market indices.

We extract objective disaster probabilities from option prices utilizing our model. This market-based assessment of disaster probability is a valuable indicator of aggregate economic conditions for practitioners, macroeconomists, and policymakers. An increase in disaster probability is associated with a decline in the conditional mean of growth—downside risks to growth vary with disaster probability while upside risks to growth remain stable when disaster probability increases. Disaster probability as registered by the financial markets contains different information about tail risks in the economy when compared to political uncertainty.

References

Adrian, Tobias, Nina Boyarchenko, and Domenico Giannone. 2019. "Vulnerable Growth." *American Economic Review*, 109 (4): 1263-89.

Andersen, T. G., Bollerslev, T., Diebold, F. X., & Labys, P. 2003. "Modeling and forecasting realized volatility," *Econometrica*, *71*(2), 579-625.

Backus, David, Mikhail Chernov, and Ian Martin. 2011. "Disasters Implied by Equity Index Options," *Journal of Finance*, 66(6), 1969–2012.

Baker, Scott, Nicholas Bloom, and Steven J. Davis. 2016. "Measuring Economic Policy Uncertainty," *Quarterly Journal of Economics*, 131(4), 1593-1636.

Bakshi, Gurdip and Georgios Skoulakis. 2010. "Do subjective expectations explain asset pricing puzzles?," *Journal of Financial Economics*, 98(3), 462-477.

Bansal, Ravi and Amir Yaron. 2004. "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59(4), 1481–1509.

Bates, D. S. 1991. "The Crash of '87: Was It Expected? The Evidence from Options Markets," *The Journal of finance*, *46*(3), 1009-1044.

Bates, D. S. 2006. "Maximum likelihood estimation of latent affine processes," *The Review of Financial Studies*, *19*(3), 909-965.

Barro, Robert J. 2006. "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 121(3), 823–866.

Barro, Robert J. 2009. "Rare Disasters, Asset Prices, and Welfare Costs," *American Economic Review*, 99(1), 243–264.

Barro, Robert J. and Tao Jin. 2011. "On the Size Distribution of Macroeconomic Disasters," *Econometrica*, 79(5), 1567–1589.

Barro, Robert J. and Jose F. Ursúa. 2008. "Macroeconomic Crises since 1870," *Brookings Papers on Economic Activity*, 255–335.

Barro, Robert. J. and Jose F. Ursúa. 2012. "Rare Macroeconomic Disasters," *Annual Review of Economics*, 4, 83–109.

Bates, David S. 1991. "The Crash of '87: Was It Expected? The Evidence from Options Markets," *Journal of Finance,* 46(3), 1009–1044.

Bollerslev, Tim, George Tauchen, and Hao Zhou. "Expected stock returns and variance risk premia." The Review of Financial Studies 22.11 (2009): 4463-4492.

Bollerslev, Tim and Viktor Todorov. 2011a. "Estimation of Jump Tails," *Econometrica*, 79 (6), 1727–1783.

Bollerslev, Tim and Viktor Todorov. 2011b. “Tails, Fears, and Risk Premia,” *Journal of Finance*, 66(6), 2165–2211.

Brenner, Menachem and Marti G. Subrahmanyam. 1988. “A Simple Formula to Compute the Implied Standard Deviation,” *Financial Analysts Journal*, 44, 80–83.

Campbell, John Y. and John H. Cochrane. 1999. “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior,” *Journal of Political Economy*, 107(2), 205-251.

Chodorow-Reich, Gabriel, Loukas Karabarbounis, and Rohan Kekre. 2019. “The macroeconomics of the Greek Depression,” No. w25900. National Bureau of Economic Research.

Chicago Board Options Exchange. 2014. “The CBOE Volatility Index (VIX),” available at cboe.com.

Cogley, Timothy and Thomas Sargent. 2008. “The market price of risk and the equity premium: A legacy of the Great Depression?,” *Journal of Monetary Economics*, 55(3), 454-476

Cox, John C. and Stephen A. Ross. 1976. “The Valuation of Options for Alternative Stochastic Processes,” *Journal of Financial Economics*, 3(1-2), 145-166

Epstein, Larry G. and Stanley E. Zin. 1989. “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework,” *Econometrica*, 57(4): 937–69.

Gabaix, Xavier. 2009. “Power Laws in Economics and Finance,” *Annual Review of Economics*, 1, 255–293.

Gabaix, Xavier. 2012. “Variable Rare Disasters: an Exactly Solved Framework for Ten Puzzles in Macro-Finance,” *Quarterly Journal of Economics*, 127(2), 645–700.

Gabaix, Xavier, Parameswaran Gopikrishnan, Vasiliki Plerou, and H. Eugene Stanley. 2003. “A Theory of Power Law Distributions in Financial Market Fluctuations,” *Nature*, 423, 267-270.

Gabaix, Xavier and Yannis Ioannides. 2004. “The Evolution of City Size Distributions,” in V. Henderson and J.F. Thisse, eds., *Handbook of Regional and Urban Economics*, v. 4, Amsterdam, North-Holland.

Gabaix, Xavier and Augustin Landier. 2008. “Why Has CEO Pay Increased So Much?,” *Quarterly Journal of Economics*, 123(1), 49-100.

Giovannini, Alberto, and Philippe Weil. 1989. “Risk Aversion and Intertemporal Substitution in the Capital Asset Pricing Model,” National Bureau of Economic Research, Working Paper 2824.

Hogg, Robert V and Allen T. Craig. 1965. *Introduction to Mathematical Statistics*, 2nd Edition, Macmillan, New York, 103-104.

Hull, John C. (2000). *Options, Futures, & Other Derivatives*, 4th ed., Upper Saddle River NJ, Prentice Hall.

Kelly, Bryan and Hao Jiang. 2014. “Tail Risk and Asset Prices,” *Review of Financial Studies*, 27(10), 2841–2871.

Kou, S.G. (2002). “A Jump-Diffusion Model for Option Pricing,” *Management Science,* 48(8), 1086-1101.

Londono, J. M. and Nancy R. Xu (2019). “Variance Risk Premium Components and International Stock Return Predictability,” *working paper*.

Lucas, Robert E. 1978. “Asset Prices in an Exchange Economy,” *Econometrica*, 46, 1429–1445.

Luttmer, Erzo G.J. 2007. "Selection, Growth, and the Size Distribution of Firms," *Quarterly Journal of Economics*, 122(3), 1103-1144.

Martin, Ian. 2015. “What is the Expected Return on the Market?” working paper, London School of Economics, June.

Mehra, Rajnish, and Edward C. Prescott. 1985. “The Equity Premium: A Puzzle,” *Journal of Monetary Economics*, 15, 145–161.

Merton, Robert C. 1976. “Option Pricing when Underlying Stock Returns Are Discontinuous,” *Journal of Financial Economics,* 3, 125–144.

Mitzenmacher, Michael. 2003. “A Brief History of Generative Models for Power Law and Lognormal Distributions,” *Internet Mathematics,* 1(2), 226-251.

Obstfeld, Maurice. 1994. “Evaluating Risky Consumption Paths: The Role of Intertemporal Substitutability,” *European Economic Review*, 38(7), 1471–1486.

Pareto, Vilfredo. 1897. *Cours d’Economie Politique*, v.2, Paris, F. Pichou.

Plerou, Vasiliki, Parameswaran Gopikrishnan, Xavier Gabaix, H. Eugene Stanley. 2004. “On the Origins of Power Law Fluctuations in Stock Prices,” *Quantitative Finance*, 4, C11-C15.

Rietz, Thomas A. 1988. “The Equity Risk Premium: A Solution,” *Journal of Monetary Economics*, 22(1), 117-131.

Ross, Steve. 2015. “The recovery theorem,” *The Journal of Finance*, 70(2), 615-648.

Seo, Sang Byung and Jessica A. Wachter. 2016. “Option Prices in a Model with Stochastic Disaster Risk,” working paper, University of Pennsylvania.

Santa-Clara, Pedro, and Shu Yan. "Crashes, volatility, and the equity premium: Lessons from S&P 500 options." *The Review of Economics and Statistics* 92.2 (2010): 435-451.

Siriwardane, Emil. 2015. "The Probability of Rare Disasters: Estimation and Implications," Harvard Business School, working paper 16-061.

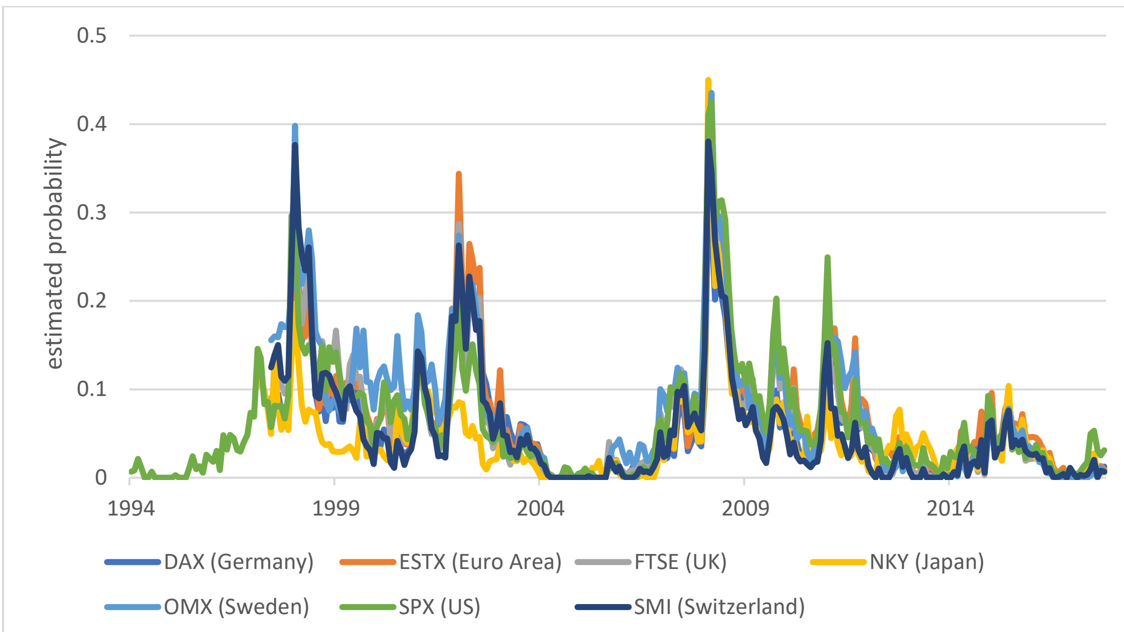
Weil, Philippe. 1990. “Nonexpected Utility in Macroeconomics,” *Quarterly Journal of Economics*, 105(1), 29–42.

Weitzman, Martin, L. 2007. "Subjective Expectations and Asset-Return Puzzles." *American Economic Review*, 97 (4): 1102-1130.

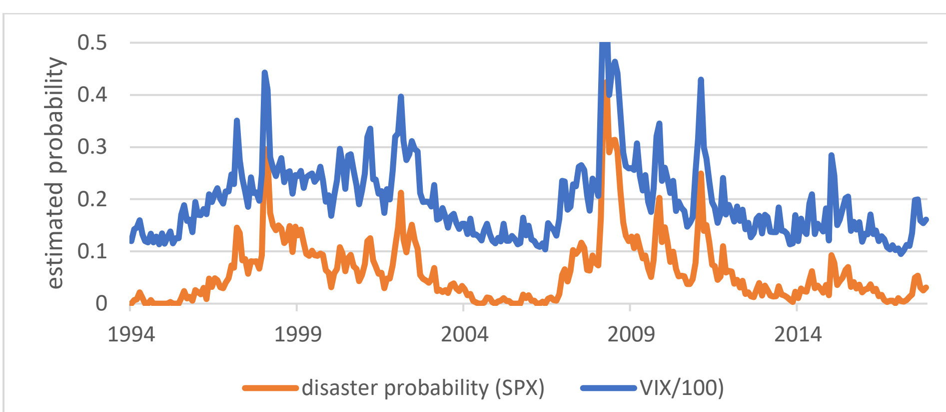
Figure 1. Estimated Disaster Probabilities

The figure shows the estimated disaster probabilities for the seven stock-market indices associated with the regressions in Table 1. The annualized disaster probability, p_{jt} for index j, is calculated from the estimated monthly fixed-effect coefficients in the form of equation (26), assuming in the formula for η_1 in equation (27) that $z_0 = 1.1$, $\gamma = 3$, and $\beta_\varepsilon = 1 + \alpha_j - \gamma$, where β_ε is given in Table 1. Panel A graphs the estimated disaster probabilities for the seven stock-market indices associated with the regressions in Table 1. Panel B is for the United States only (SPX). The VIX measure of volatility is discussed in Chicago Board Options Exchange (2014). Panel C is for the seven indices estimated jointly (last column of Table 1).

Panel A: Seven Stock Market Indices Individually



Panel B: Disaster probability and the VIX



Panel C: Seven Stock-Market Indices Jointly

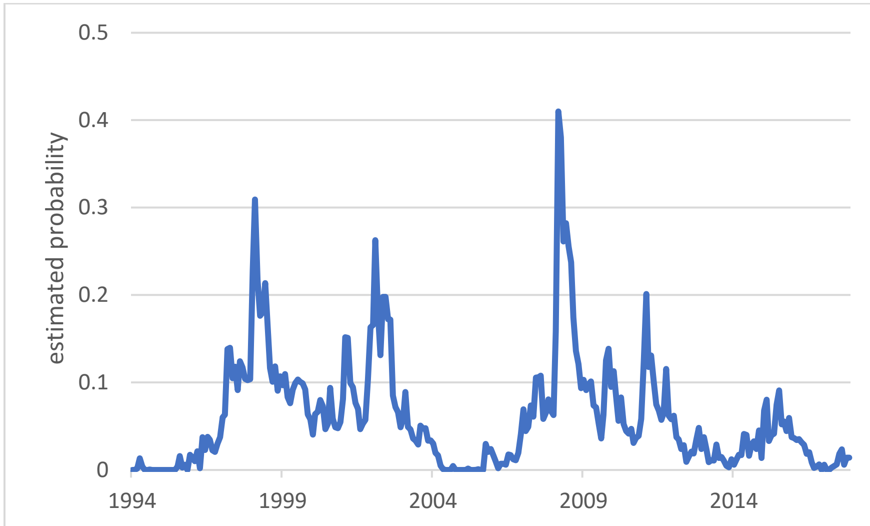


Figure 2. Estimated U.S. Disaster Probabilities, 1983-2018

This exhibit presents the estimated U.S. disaster probabilities, p_t, associated with the regression in Table 3. The underlying data from August 1994 to June 2018 are the OTC data based on the S&P 500. The data from June 1983 to July 1994 are associated with the S&P 100 and are from the Berkeley Options Data Base. The methodology for inferring disaster probabilities from the estimated monthly fixed effects corresponds to that used in Figure 1. The left panel presents the estimated probability before 1990 and the right panel presents the estimated probability since 1990.

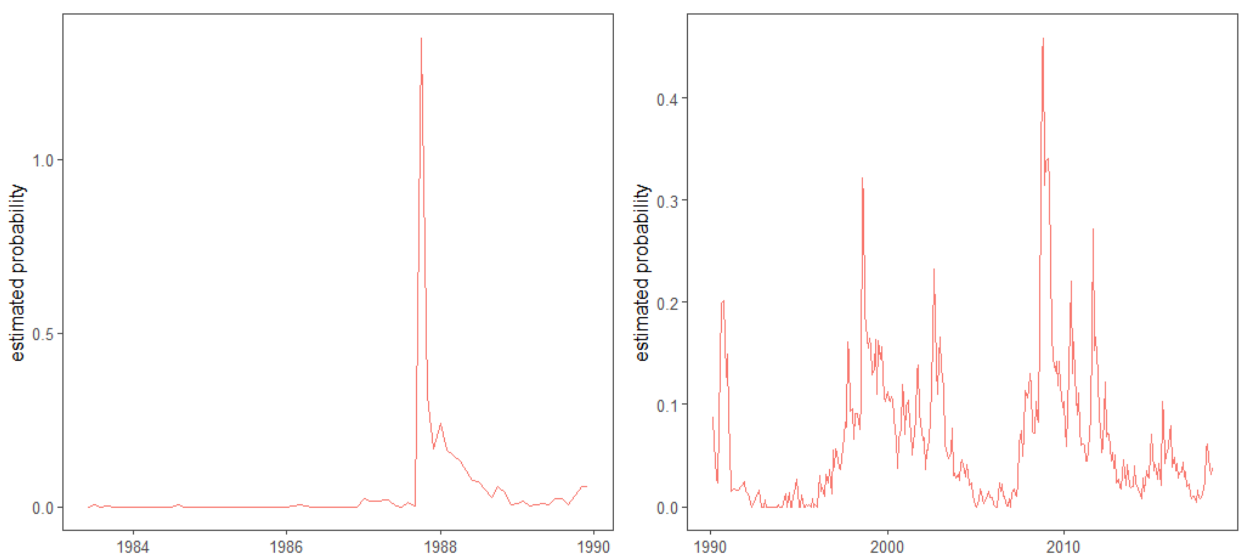


Figure 3. Forecast of Conditional Growth for United States

This figure shows the estimated coefficients in quantile regressions of one-year and one-quarter ahead U.S. real GDP growth (top and middle) and one-month ahead Industrial Production (IP) growth (bottom) on disaster probability (left) and EPU index (right). The regressions also include a lag of the growth measure of the same horizon as the forecasting period without overlap. For instance, the OLS regression for the top left panel is $\Delta GDP_{t+1} = \beta_0 + \beta_1 \Delta GDP_t + \beta_1 p_t + \epsilon$, where $\Delta GDP_{t+1} = \frac{GDP_{t+1\,yr} - GDP_t}{GDP_t}$ and p_t is the last monthly value observed before t. The red solid lines graph the coefficients of the forecasting regressions for different quantiles (τ). The dotted blue lines are the OLS estimates. The 95-percentile and 90 percentile confidence intervals (grey bands) associated with the OLS estimates are constructed with Newey-West adjusted standard errors. The sample period is 1983 to 2018 (the extended U.S. disaster probability time series). GDP growth measures are the year-over-year percentage changes and the quarter-over-quarter seasonally adjusted annual rate, both observed quarterly. IP growth is the change in the log of the IP level observed monthly.

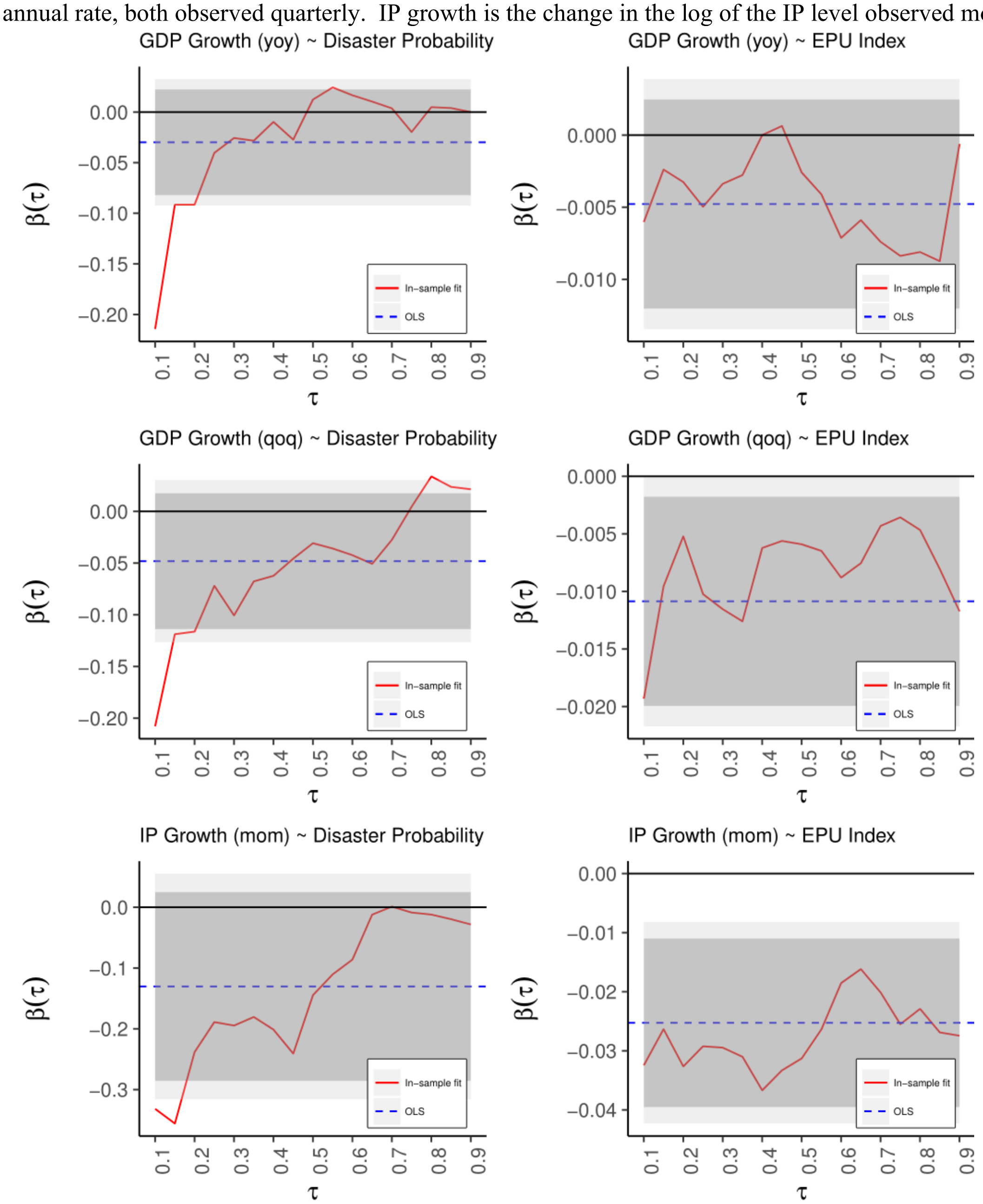
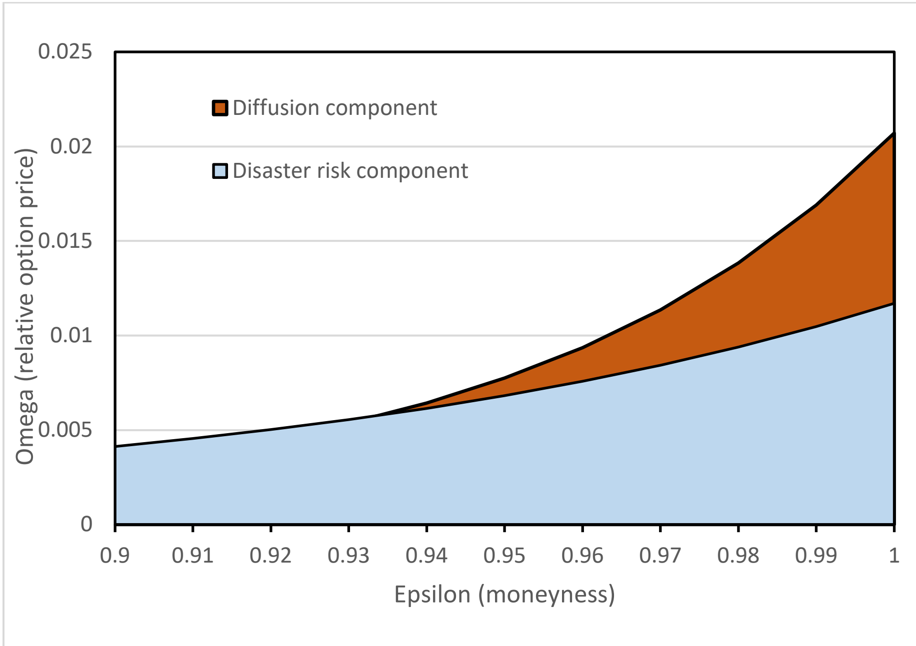


Figure 4. Decomposition of Option Prices into Disaster-Risk and Diffusion Components



This figure shows the decomposition of put options prices into components that represent disaster risk and a diffusion process. To obtain the contribution of disaster risk, we calculate the fitted values of near-the-money put prices (epsilon (moneyness) between 0.9 and 1) using our model as applied to the overall sample of countries from Table 1. The disaster-risk component of the graph shows the application of these estimates to short maturity options ($T = 1\ month$) to averages of data for the U.S. S&P 500 from 1994 to 2018. The estimated diffusion component is the difference between the observed options prices and the fitted values.

Table 1 Regressions for Put-Options Prices

OTC Data, 1994-2018

Index	**SPX**	**FTSE**	**ESTX**	**DAX**	**NKY**	**OMX**	**SMI**	**--**
Country/Region	**US**	**UK**	**EURO**	**GER**	**JAP**	**SWE**	**SWZ**	**ALL**
	Aug94-Jun18	Jan98-Jun18	Jun98-Jun18	Jan00-Jun18	Sep 97-Jun18	Jan98-Jun18	Jan98-Jun18	Aug94-Jun18
$\boldsymbol{\beta_T}$	0.992	0.997	0.943	0.946	0.881	0.922	1.004	0.961
	(.040)	(.046)	(.045)	(.044)	(.032)	(.045)	(.046)	(.042)
$\boldsymbol{\beta_\epsilon}$	4.73	4.66	4.45	4.15	4.01	4.64	4.75	4.55
	(.465)	(.429)	(.463)	(.452)	(.480)	(.460)	(.428)	(.447)
$\boldsymbol{\alpha^* - \alpha}$	9.42	8.29	7.87	7.68	10.87	8.83	9.71	9.35
	(6.594)	(6.101)	(5.386)	(4.572)	(5.990)	(6.904)	(6.190)	(6.284)
$\boldsymbol{\eta_2 q}$	0.087	0.078	0.095	0.102	0.128	0.103	0.090	0.098
	(.044)	(.033)	(.034)	(.032)	(.064)	(.051)	(.044)	(.046)
Implied est of $\boldsymbol{\alpha}$	6.73	6.66	6.45	6.15	6.01	6.64	6.75	6.55
Implied est of $\boldsymbol{\eta_1}$	0.725	0.737	0.777	0.844	0.882	0.740	0.720	0.756
R-squared	0.973	0.968	0.970	0.968	0.957	0.957	0.965	0.914
$\boldsymbol{\sigma}$	0.0012	0.0014	0.0016	0.0018	0.0019	0.0021	0.0014	0.0026
$\boldsymbol{N}$	5740	4920	4820	4920	4968	4920	4920	35208
Mean dep var.	0.0038	0.0040	0.0052	0.0054	0.0048	0.0053	0.0034	0.0045
$\boldsymbol{\sigma}$ dep var.	0.0075	0.0080	0.0095	0.0099	0.0092	0.0101	0.0073	0.0089

This table presents non-linear least-squares regression estimates of the model for pricing far-out-of-the money put options with variable disaster probability. We use OTC data on relative put-option prices, Ω, for seven stock-market indices with maturity, T, of 30, 60, 90, and 180 days and relative exercise price, ε, of 0.5, 0.6, 0.7, 0.8, and 0.9. The estimation corresponds to equation (26): $\Omega = \mathrm{T}\varepsilon^{1+\alpha-\gamma} \cdot [\eta_1 p_t + \eta_2 q \varepsilon^{(\alpha^*-\alpha)}]$, where p_t is the disaster probability, α is the tail parameter for disaster sizes, α^* is the tail parameter for stock-price changes induced by upward jumps in p_t, q is the probability of an upward jump in p_t, γ is the coefficient of relative risk aversion, and η_1 and η_2 are constants shown in equation (27). We use the estimated monthly fixed effects for each stock-market index to gauge the variations in $\eta_1 p_t$ and then use a calibrated value of η_1 to infer levels of p_t. The results are in Figure 1. The estimation constrains $p_t \geq 0$ for each observation. This constraint turns out to be binding on average for 8% of the observations for the seven stock-market indices. The column labeled "all" pools the data on the seven stock-market indices and uses the same coefficients and set of monthly fixed effects for all indices. The estimated exponent on T, β_T, should equal 1. The estimated exponent on the first ε term, β_ϵ, should equal $1 + \alpha - \gamma$. Implied estimates of α are shown, based on $\gamma = 3$. Implied estimates of η_1 are shown, assuming in equation (27) that the threshold value for disaster size is $z_0 = 1.1$. Cross-section-clustered standard errors (which allow for serial correlation of the error terms) are in parentheses.

Table 2. Statistics for Estimated Disaster Probabilities

Index	sample	mean	std. dev.	maximum
SPX (US)	1994.08-2018.06	0.062	0.066	0.425
FTSE (UK)	1998.01-2018.06	0.065	0.070	0.398
ESTX (Euro area)	1998.06-2018.06	0.069	0.068	0.371
DAX (Germany)	2000.01-2018.06	0.058	0.064	0.324
NKY (Japan)	1998.01-2018.06	0.045	0.052	0.450
OMX (Sweden)	1998.01-2018.06	0.077	0.078	0.435
SMI (Switz.)	1998.01-2018.06	0.056	0.070	0.380
ALL	1994.08-2018.06	0.061	0.065	0.410

This table presents statistics on estimated disaster probabilities from the regressions in Table 1. The disaster probabilities are calculated as described in the notes to Figure 1.

Table 3. Regression for U.S. Put-Options Prices

Berkeley and OTC Data, 1983-2018

$\boldsymbol{\beta_T}$	0.992 (0.036)
$\boldsymbol{\beta_\epsilon}$	4.91 (0.49)
$\boldsymbol{\alpha^* - \alpha}$	10.61 (8.27)
$\boldsymbol{\eta_2 q}$	0.091 (0.059)
R-squared	0.964
$\boldsymbol{\sigma}$	0.0015
$\boldsymbol{N}$	6370

Statistics for Estimated Disaster Probabilities (shown in Figure 2)

Period start	Period end	mean	std. dev.	maximum
June 1983	**June 2018**	0.064	0.097	1.35
June 1983	**Sept 1987**	0.004	0.007	0.025
Oct 1987	**Sept 1988**	0.283	0.410	1.35
Oct 1988	**Jul 1994**	0.029	0.047	0.202
Aug 1994	**June 2018**	0.070	0.071	0.459

The form of the regression corresponds to that for the U.S. SPX in Table 1. The data from June 1983 to July 1994 are based on the S&P 100 index and are market-based information from the Berkeley Options Data Base. The data from August 1994 to June 2018 are OTC values based on the S&P 500, as in Table 1. For the Berkeley data, we formed monthly panels of put-options prices by aggregating quotes from the last five trading days of each month. We applied a bivariate linear interpolation on the implied volatility surface to obtain put prices with granular strikes at every 10% moneyness interval and maturities ranging from one to six months. The methodology for inferring disaster probabilities from the estimated monthly fixed effects corresponds to that used in Figure 1, with the results shown in Figure 2. Because of missing information in the Berkeley data, many months before August 1994 do not appear in the regression or in Figure 2. Cross-section-clustered standard errors (which allow for serial correlation of the error terms) are in parentheses.

Table 4 Regression for U.S. Put-Options Prices OptionMetrics Data, 1996-2017

$\boldsymbol{\beta_T}$	0.951 (0.033)
$\boldsymbol{\beta_\epsilon}$	4.53 (0.43)
$\boldsymbol{\alpha^* - \alpha}$	8.98 (6.09)
$\boldsymbol{\eta_2 q}$	0.087 (0.039)
R-squared	0.973
$\boldsymbol{\sigma}$	0.00144
$\boldsymbol{N}$	3886

This regression corresponds to that for the U.S. SPX in Table 1, except for the use of market-based OptionMetrics data over the period January 1996-December 2017.

Table 5 Model Fit with Maturity-dependent Epsilon Thresholds

	A. Regular sample	**B. Expanded sample**
$\boldsymbol{\beta_T}$	1.12 (0.083)	1.14 (0.042)
$\boldsymbol{\beta_\epsilon}$	5.10 (0.58)	5.44 (0.31)
$\boldsymbol{\alpha^* - \alpha}$	6.10 (6.37)	5.94 (6.68)
$\boldsymbol{\eta_2 q}$	0.068 (0.059)	0.050 (0.028)
R-squared	0.959	0.963
$\boldsymbol{\Sigma}$	0.000829	0.000751
$\boldsymbol{N}$	4879	17507

This table presents the regression results using put options with the upper threshold for epsilon (moneyness) that varies with the maturity of the option. Column A presents the result using option prices with the upper epsilon threshold of 0.9 for the one-month horizon and 0.8 for maturities beyond one month. Column B shows the result of an expanded sample. We obtain more granular option prices at each epsilon interval of 0.025 through interpolating the implied volatility surface. We then calibrate the upper epsilon threshold at each maturity such that a Brownian diffusion process with monthly volatility of 0.035 (annualized volatility of 12%) breaches the thresholds with only a small chance (less than 1%). This procedure results in an epsilon cutoff of 0.9 for options of one-month maturity and decreases for each 0.025 step down in epsilon, and the epsilon cutoff at the 6-month horizon is 0.80.

Table 6

Stability over Different Samples of Estimated Coefficients in Regressions for Put-Options Prices

Coefficient	1994.08-2003.01	2003.02-2008.02	2008.03-2013.03	2013.04-2018.06	p-value
	U.S. (SPX)				
$\boldsymbol{\beta_T}$	0.998	1.203	0.920	1.303	0.000
	(.043)	(.048)	(.042)	(.057)	
$\boldsymbol{\beta_\epsilon}$	5.12	6.30	4.00	5.63	0.000
	(.47)	(.59)	(.48)	(.66)	
$\boldsymbol{\alpha^* - \alpha}$	10.65	6.03	4.92	5.06	0.055
	(8.46)	(3.92)	(3.15)	(3.29)	
$\boldsymbol{\eta_2 q}$	0.098	0.080	0.090	0.072	0.000
	(.062)	(.015)	(.017)	(.015)	
	All (seven stock-market indices)				
$\boldsymbol{\beta_T}$	0.949	1.117	0.911	1.148	0.000
	(.046)	(.047)	(.041)	(.042)	
$\boldsymbol{\beta_\epsilon}$	4.67	6.05	3.66	5.74	0.000
	(.57)	(.57)	(.44)	(.60)	
$\boldsymbol{\alpha^* - \alpha}$	9.26	6.06	5.44	7.31	0.000
	(8.04)	(4.32)	(2.69)	(4.34)	
$\boldsymbol{\eta_2 q}$	0.111	0.085	0.100	0.077	0.000
	(.059)	(.017)	(.016)	(.020)	

Note: The estimated coefficients shown for four sub-periods correspond to those shown for full samples in Table 1. The sub-periods were chosen to have roughly equal numbers of observations, starting from January 1998, by which the data are available for five of the seven stock-market indices considered in Table 1. The results apply to the U.S. (SPX) stock-market index and for the pooled sample of all seven stock-market indices. The p-values are for the hypothesis that the associated coefficient is the same across the four sub-periods. The p-value for the joint hypothesis that all coefficients are equal across the sub-periods has a p-value of 0.000 for the U.S. SPX and for the pooled sample of all seven stock-market indices.